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Existence theorems for some abstract nonlinear non-autonomous systems with delays ☆,☆☆

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ABSTRACT

For some abstract classes of nonlinear non-autonomous systems with variable and state-dependent delays existence, non-existence and multiplicity of periodic solutions are discussed. To illustrate the efficiency of the method, we obtain some well-known results for applied systems as corollaries of our existence theorems.

1. Introduction

Lotka–Volterra delayed systems are extensively used to model prey-predator population dynamics. For example, the system

$$x_i'(t) = x_i(t)\left[c_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t-\tau_i(t))\right] \tag{1.1}$$

was under study in [9,19,21,26,28,34]; Gilpin-Ayala model

$$x_i'(t) = x_i(t)\left[c_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j^{\theta_j}(t-\tau_{ij}(t))\right] \tag{1.2}$$

in [8,9]; logarithmic Lotka–Volterra

$$x_i'(t) = x_i(t)\left[a_i(t) - \sum_{j}^{n} b_{ij}(t)\ln x_j(t) - \sum_{j}^{n} c_{ij}(t)\ln x_j(t-\tau_{ij}(t))\right] \tag{1.3}$$

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in [43]; Hopfield neuron network models

$$x_i'(t) = -a_i(t)x_i(t) + \sum_j^n a_{ij}(t)f_{ij}(x(t)) + \sum_j^n b_{ij}(t)f_{ij}(x(t - \tau_{ij}(t))) \tag{1.4}$$

were studied in [7,13,16,25,29,44].

In [15,30,36,39] the following general models for an n-dimensional vector $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ were under investigation

$$x'(t) = A(t)x(t) + \lambda f(t, x(t - \tau(t))), \tag{1.5}$$

$$x'(t) = A(t, x(t))x(t) + f(t, x(t - \tau)) \tag{1.6}$$

and

$$x'(t) = \nabla g(x(t)) + f(t, x(t - \tau)). \tag{1.7}$$

Here, $A(t,x)$ is a continuous $n \times n$ matrix, $f(t,x)$ is a continuous n-dimensional vector function and $g(x)$ is a C^1 scalar function.

There have been various approaches developed to examine the existence of periodic solutions for delay differential equations since the first study published by Browder in 1962, such as fixed point theorems, Hopf bifurcation theorems, Poincaré–Bendixson theorems, Lyapunov functions, the spectral theory of matrices, Morse theory, Galerkin methods and coincidence degree theory (see, for example, [1–4,10,11,27]). Some interesting results were recently obtained in [17,22,24,32,33,39,40,42]. Multiple systems of population dynamics were recently studied in [3,5,6,18,20,23,35,37,38,41].

Motivated by these models, we introduce and study the most general system and discuss its applications. Some new and interesting sufficient conditions are obtained to guarantee the existence, non-existence and multiplicity of periodic solutions.

2. Continuation theorem for the abstract model and applications

Let

$$X := \{x \in C(\mathbb{R}, \mathbb{R}^n) : x(t + \omega) = x(t) \quad \text{for all } t\}$$

and consider the functional differential equation

$$x'(t) = \Phi(x)(t), \tag{2.1}$$

where $\Phi : X \to X$ is continuous and maps bounded sets into bounded sets. For $x \in X$, its absolute maximum and minimum values and its average $\frac{1}{\omega}\int_0^\omega x(t)\,dt$ are denoted by x_{max}, x_{min} and $\bar{x}$, respectively. The euclidian norm of a vector $y \in \mathbb{R}^n$ shall be denoted by $|y|$. Let U be an open and bounded subset of X and denote its closure by $cl(U)$. If $\mathscr{K} : cl(U) \to X$ is compact with $\mathscr{K}u \neq u$ for $u \in \partial U$, then the Leray–Schauder degree of the Fredholm operator $\mathscr{F} = Id - \mathscr{K}$ at 0 shall be denoted by $deg_{LS}(\mathscr{F}, U, 0)$ (for a detailed definition and properties of the degree see for example [27]). Finally, we identify the subset of constant functions of X with $\mathbb{R}^n$; thus, a vector $\gamma \in \mathbb{R}^n$ may be interpreted as an element of X so the function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\phi(\gamma) := \overline{\Phi(\gamma)} = \frac{1}{\omega}\int_0^\omega \Phi(\gamma)(t)\,dt$$

is well defined. The following continuation theorem will be the key for further studies. The proof follows essentially the same outline of analogous results (see e. g. [10]) so it is omitted.

Theorem 2.1. *Assume there exists a bounded open subset $U \subset X$ such that*

1. *If $x'(t) = \lambda\Phi(x)(t)$ for some $x \in cl(U)$ and $0 < \lambda < 1$, then $x \in U$.*
2. *$\phi(x) \neq 0$ for $x \in \partial U \cap \mathbb{R}^n$.*
3. *$deg_B(\phi, U \cap \mathbb{R}^n, 0) \neq 0$ (deg_B stands for the Brouwer degree).*

Then (2.1) has at least one solution $x \in cl(U)$.

Consider the system

$$x'(t) = F(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_m)) \tag{2.2}$$

with $F : \mathbb{R} \times \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$ continuous and ω-periodic in t and $\tau_j = \tau_j(t, x(t))$, with $\tau_j : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ continuous, positive and ω-periodic in t. For convenience, given an arbitrary bounded open set $\Omega \subset \mathbb{R}^n$, we define

$$X_\Omega := \{x \in X : x(t) \in \Omega \quad \text{for all } t\}.$$