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Computing topological entropy for periodic sequences of unimodal maps

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ABSTRACT

In this paper we introduce an algorithm which allows us to compute the topological entropy of a class of piecewise monotone continuous interval maps. The algorithm can be applied to a class of economic models called duopolies, and it can be useful to compute the topological entropy of periodic sequences of continuous maps which have been used in some population growth models.

1. Introduction

Recently, dynamical systems given by periodic sequences of maps has been analyzed by several authors (see e.g. [2–7,17,22–25] among others). This interest has been motivated by models in population dynamics (see e.g. [20,26]), and economic dynamics, the so-called duopolies (see e.g. [10,11,21,28,31,37,36]). Let us pay the attention on the last kind of models.

Sometimes, a duopoly can be modeled by a two–dimensional map $F(x,y) = (f(y), g(x)), (x,y) \in [a,b] \times [c,d]$, where $[a,b]$ and $[c,d]$ are compact intervals of the non–negative real line and f and g are called reaction functions. It was proved in [21] that

$$F^2(x,y) = ((f \circ g)(x), (g \circ f)(y))$$

and therefore the dynamics of F can be studied from the one–dimensional maps $f \circ g$ and $g \circ f$. In particular, the topological entropy of F (see e.g. [1] or [15] for definition) can be computed by the formula $h(F) = h(f \circ g) = h(g \circ f)$. In addition, it was proved in [27] that the topological entropy of the periodic sequence of maps $f_{1,\infty} = (f, g, f, g, \ldots)$ is computed by $h(f_{1,\infty}) = \frac{1}{2} h(f \circ g)$.

So, the analysis of the dynamics of periodic sequences of maps or duopoly models seems to be easy: just consider the composition and work with it. However, if we consider particular duopoly models [28,31,37,36] we check that maps f and g are unimodal (with two strictly monotone pieces) and therefore $f \circ g$ can have 4 monotone pieces. As a consequence, an efficient computation of the topological entropy is not an easy task.

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The algorithm that we present in this paper solves the above problem, that is, we are able to compute with prescribed accuracy the topological entropy of the duopoly models mentioned above and, in general, for any duopoly model which might be described in terms of unimodal maps. In addition, we could also do the same for some population dynamics models described by periodic sequences of unimodal maps.

It is well–known that positive topological entropy of continuous maps implies Li–Yorke chaos (see e.g. [12,29] for definition). In addition, the regularity conditions of the above mentioned models can imply that nonwandering intervals do not exist, avoiding the existence of Li–Yorke chaotic maps with zero topological entropy in these models (see [9,30,39]). In other words, topological entropy can be used as a topological dichotomy between order and chaos. Therefore, the algorithm that we present here is useful to prove the existence of chaos in a rigorous way.

Let us recognize that the definition of topological entropy makes it extremely difficult to compute it in a practical way (we refer the reader to [33], where Abel prize John Milnor wonders whether topological entropy can be computed efficiently). For unimodal maps the question of computing topological entropy was investigated by several authors who proposed algorithms to compute it (see [13,16,19]). In [14], an algorithm for computing the topological entropy of bimodal maps with prescribed accuracy is presented as an extension of the ideas of [13]. When the number of monotone pieces of a map increases, the computation of topological entropy becomes very hard. Let us mention that in [40], Steinberger presented a method for computing the topological entropy of a piecewise monotone map by means of algebraic tools as transition matrices, although its implementation for families depending on several parameters seems not to be simple. Recently, a general algorithm is presented in [8], although without any error estimations. Finally, in [38], the computation of topological entropy of the composition of two logistic map was done with an algorithm inspired in those given in [13,14].

Our approach is simpler and also borrows ideas from Block and Keesling's algorithm in [14], but only allows us to compute the topological entropy for families of piecewise monotone maps with the additional property that all maxima (resp. minima) have the same value. This condition is restrictive, but it is the case of the composition of two unimodal maps. In particular, we give the specific algorithm to compute the topological entropy of a continuous piecewise monotone map with four monotone pieces such that the cardinality of the set of the values attained by the local extremum is equal to two, and we use it to compute the topological entropy of two duopoly models introduced in [28,31]. Let us remark that the algorithm can be easily modified to be applied to piecewise monotone maps with more than 4 monotone pieces, e.g. two maxima and two minima with the same forward image, respectively. In the sequel, we will make a more precise explanation.

Definition 1. Let $I = [a, b]$ be a compact interval, $a, b \in \mathbb{R},\ a < b$. A continuous map $f : I \to I$ is called *piecewise monotone* if and only if there exist points

$$a = t_0 < t_1 < \cdots < t_n = b$$

such that for each $i = 0, \ldots, n-1, f$ is either strictly increasing or strictly decreasing on $[t_i, t_{i+1}]$. The points $t_1, \ldots, t_{n-1}$ are called *turning points* of f. We denote by $\mathcal{P}(I)$ the family of continuous piecewise monotone maps defined on the interval I.

As a first approach, we can compute the topological entropy of a piecewise monotone map by using the Misiurewicz–Szlenk Theorem (see [35,34]), which gives us the characterization of the *topological entropy* for a continuous piecewise monotone map f in terms of the number of monotone pieces of f^n (here $f^1 = f$ and $f^n = f^{n-1} \circ f$ for $n > 1$) as follows.

Theorem 1. *Let $f : I \to I$ be a continuous piecewise monotone map. Then the topological entropy of $f, h(f)$ is given by*

$$h(f) = \lim_{n \to +\infty} \frac{1}{n} \log c(f^n),$$

where $c(f)$ denotes the number of pieces of monotonicity of f.

However, the above formula is not good from a practical point of view, and additional tools are necessary to compute the topological entropy of piecewise monotone maps.

Definition 2. Let $I = [a, b]$ and $f \in \mathcal{P}(I)$, we denote by $Ext(f) \subset (a, b)$ the set of the turning points of f. For $N \in \mathbb{N},\ N > 2$, we define the class of N-piecewise continuous maps given by

$$\mathcal{C}_N(I) = \{f \in \mathcal{P}(I) : Card(Ext(f)) = N - 1 \text{ and } Card(f(Ext(f))) = 2\}. \tag{1}$$

We denote by

$$\mathcal{C}(I) = \bigcup_{N \in \mathbb{N}} \mathcal{C}_N(I). \tag{2}$$

Observe that the graph of a piecewise monotone map f in the class $\mathcal{C}(I)$ has the form given in Fig. 1.

Our aim in this paper is to compute the topological entropy of maps in the class $\mathcal{C}_N(I)$, for $I = [a, b]$, a closed interval. For that, there is no loss of generality in assuming that the maps that we are considering take endpoints of the interval I (the points $\{a, b\}$) to endpoints. The reason is that we can replace the map f by another map g in the same class having the same number of monotone pieces and the same topological entropy with g taking the endpoints of I to endpoints.

Let $I = [a, b]$ be a closed interval and $f \in \mathcal{C}_N(I)$, then let