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Approximate analytical solution for the fractional modified KdV by differential transform method

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ABSTRACT

In this paper, the fractional modified Korteweg-de Vries equation (fmKdV) and fKdV are introduced by fractional derivatives. The approach rest mainly on two-dimensional differential transform method (DTM) which is one of the approximate methods. The method can easily be applied to many problems and is capable of reducing the size of computational work. The fractional derivative is described in the Caputo sense. Some illustrative examples are presented.

1. Introduction

Fractional differential equations are studied in various fields of physics and engineering. The numerical and analytical approximations of such problems have been intensively studied since the work of Padovan[1]. Recently, several mathematical methods including the Adomian decomposition method [4,5] variational iteration method [6,7] homotopy analysis method [18,20,21] and fractional method [2] have been developed to obtain exact and approximate analytic solutions. Among these solution techniques, the variational iteration method and the Adomian decomposition method are the most clear methods of solution of fractional differential and integral equations, because they provide immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to nonlinear differential equations without linearization or discretization.

In this paper, we consider the generalized KdV equation of the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \varepsilon(u(x,t))^m \frac{\partial^\beta u(x,t)}{\partial x^\beta} + v\frac{\partial^3 u(x,t)}{\partial x^3} = g(x.t). \tag{1.1}$$

For $t > 0, 0 < \alpha, \beta \leqslant 1$, where ε, v are constants, $m = 0, 1, 2$ and α and β are parameters describing the order of the fractional time and space-derivatives. If $m = 0, m = 1$ and $m = 2$, Eq. (1.1) becomes the linear fractional KdV, nonlinear fractional KdV and fractional modified KdV (fmKdV), respectively. The function $u(x,t)$ is assumed to be a causal function of time and space. The fractional derivatives are considered in Caputo sense. In case of $\alpha = \beta = 1$, Eq. (1) reduces to the classical mKdV. İn this paper the application of DTM will be extended to obtain approximate solutions of fmKdV and fKdV ($m = 1$ and $m = 2$).

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In the last decades, fractional calculus has found diverse applications in various scientific and technological fields [2,3], such as thermal engineering, acoustics, fluid mechanics, biology, chemistry, electromagnetism, control, robotics, diffusion, edge detection, turbulence, signal processing and many other physical processes.

The differential transform method was first applied in the engineering domain in [8]. The differential transform method is a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation. However, the differential transform method obtains a polynomial series solution by means of an iterative procedure. Recently, the application of differential transform method is successfully extended to obtain analytical approximate solutions to ordinary differential equations of fractional order [9]. Application of fractional calculus in physics was presented in [3]. A comparison between the variational iteration method and Adomian decomposition method for solving fractional differential equations is given in [11]. Very recently, Hashim[12] demonstrated the application of homotopy-perturbation method for solving fmKdV.

2. Fractional calculus

There are several definitions of a fractional derivative of order $\alpha > 0$ [2,22].e.g.Riemann–Liouville, Grunwald–Letnikow, Caputo and Generalized Functions Approach The most commonly used definitions are the Riemann–Liouville and Caputo. We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1. A real function $f(x), x > 0$, is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty)$, and it said to be in the space C_μ^m iff $f^m \in C_\mu, m \in N$.

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha \geqslant 0$, of a function $f \in C_\mu, \mu \geqslant -1$, is defined as

$$J_0^\upsilon f(x) = \frac{1}{\Gamma(\upsilon)} \int_0^x (x-t)^{\upsilon-1} f(t)dt, \quad \upsilon > 0,$$
$$J^0 f(x) = f(x).$$

It has the following properties:

For $f \in C_\mu,\ \mu \geqslant -1,\ \alpha,\ \beta \geqslant 0$ and $\gamma > 1$:

1. $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,
2. $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,
3. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Riemann–Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for the physical problems of the real world since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations.

Definition 2.3. The fractional derivative of $f(x)$ in the caputo sense is defined as

$$D_*^\upsilon f(x) = J_a^{m-\upsilon} D^m f(x) = \frac{1}{\Gamma(m-\upsilon)} \int_0^x (x-t)^{m-\upsilon-1} f^{(m)}(t)dt, \text{ for } m-1 < \upsilon < m, \quad m \in N,\ x > 0,\ f \in C_{-1}^m.$$

Lemma 2.1. *If* $m-1 < \alpha < m, m \in N$ *and* $f \in C_\mu^m, \quad \mu \geqslant -1$, *then*

$$D_*^\alpha J^\alpha f(x) = f(x),$$
$$J^\alpha D_*^\upsilon f(x) = f(x) - \sum_{k=0}^{m-1} f^k(0^+)\frac{x^k}{k!}, \quad x > 0.$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem. In this paper, we have considered fmKdV and fKdV, where the unknown function $u = u(x,t)$ is a assumed to be a causal function of fractional derivatives are taken in Caputo sense as follows:

Definition 2.4. For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_{*t}^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\xi)^{m-\alpha-1} \frac{\partial^m u(x,\xi)}{\partial \xi^m} d\xi, & \text{for } m-1 < \alpha < m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha = m \in N. \end{cases}$$