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On the solutions of time-fractional reaction–diffusion equations

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ABSTRACT

In this paper, a new application of generalized differential transform method (GDTM) has been used for solving time-fractional reaction–diffusion equations. To illustrate the reliability of the method, some examples are provided.

1. Introduction

The nonlinear reaction–diffusion equations have found numerous applications in pattern formation in many branches of biology, chemistry, and physics [1–4]. Reaction–diffusion (RD) equations have also applied to numerous other problems which appeal in the targeted scientific community [5–18]. For example, the RD equations are employed to describe the CO oxidation on Pt(1 1 0) [10], the study of Ca^{2+} waves on Xenopus oocytes [14], and the study of reentry in heart tissue [17,18].

There has been a wide variety of numerical methods, e.g., finite difference techniques, finite element methods, spectral techniques, adaptive and non-adaptive algorithms, etc., which have been developed for its numerical solution [19,20]. Amongst the most recent numerical techniques, it is worth mentioning on-standard finite difference methods [21,22], hybrid boundary integral procedures [23], the nodal integral scheme [24] and piecewise hybrid analytical numerical algorithms [25].

In recent years, there has been a great deal of interest in fractional reaction–diffusion (FRD) systems [26–35] which from one side exhibit self-organization phenomena and from the other side introduce a new parameter to these systems, which is a fractional derivative index, and it gives a greater degree of freedom for diversity of self-organization phenomena. At the same time, the process of analyzing such FRD systems is much more complicated from the analytical and numerical point of view.

Trapping reactions between molecules embedded in biological samples and disordered materials are usually handicapped by the porous and statistical fractal structure of these media [36]. In some cases this gives rise to subdiffusion the particles, i.e., the mean square displacement $\langle r^2(t)\rangle$ of the particles from the original starting site is no longer linear on time, but verifies a generalized Fick's second law:

$$\langle r^2(t)\rangle \approx \frac{2K_\alpha}{\Gamma(1+\alpha)} t^\alpha \tag{1.1}$$

where $0 < \alpha < 1$ is the (anomalous) diffusion exponent and K_α is the diffusion coefficient. There are many other instances in which subdiffusion processes appear [37–44].

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In this paper, we present a solution of a more general model of reaction–diffusion equations:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = D\frac{\partial^2 u}{\partial x^2} + f(u) \qquad 0 < \alpha \leqslant 1 \quad t > 0 \quad x \in \Re \tag{1.2}$$

where D is the diffusion coefficient and $f(u)$ is a nonlinear function representing reaction kinetics. It is interesting to observe that for $f(u) = 6u(1-u)$, Eq. (1.2) reduces to the time-fractional Fisher equation which was originally proposed by Fisher [45] as a model for the spatial and temporal propagation of a virile gene in an infinite medium. It is encountered in chemical kinetics [46], flame propagation [47], auto catalytic chemical reaction [48], nuclear reactor theory [49], neurophysiology [50], and branching Brownian motion process [51]. Recently, Fisher equation combines diffusion with logistic non-linearity and concludes problems such as nonlinear evolution of a population in a one-dimensional habitat. If we set $f(u) = u(1-u)(u-\mu)$, it gives rise to the time-fractional Fitzhugh–Nagumo equation, which is an important nonlinear reaction–diffusion equation and applied to model the transmission of nerve impulses [52,53], also used in biology and the area of population genetics in circuit theory [54]. When $\mu = -1$, the Fitzhugh–Nagumo equation reduces to the real Newell–Whitehead equation.

We use the generalized differential transform method (GDTM), for solving nonlinear fractional reaction–diffusion partial differential equations, which was presented by [55–57]. This method is based on differential transform method (DTM) [58–61], The DTM introduces a promising approach for many applications in various domains of science. By using the DTM, we obtain a series solution, actually a truncated series solution. This series solution does not exhibit the real behaviors of the problem but gives a good approximation to the true solution in a very small region. Odibat et al. [62] proposed a reliable algorithm of the DTM. The new algorithm accelerates the convergence of the series solution over a large region and improve the accuracy of the DTM. The validity of the modified technique is varied through illustrative examples of Lotka-Volterra, Chen and Lorenz systems. The GDTM method also based on generalized Taylor's formula [63], and Caputo fractional derivatives, which are defined as [64]:

$$D^\alpha f(x) = I^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha+1} f^{(m)}(t)\, dt$$

For $m-1 < \alpha \leqslant m,\ m \in N,\ x > 0$; for the Caputo derivative we have:

$$D^\alpha C = 0, \quad (C \text{ is a constant}), \qquad D^\alpha t^n = \begin{cases} 0, & (n \leqslant \alpha - 1) \\ \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} t^{n-\alpha}, & (n > \alpha - 1) \end{cases}$$

For m being the smallest integer that exceeds α, the Caputo fractional derivatives of order $\alpha > 0$ are defined as:

$$D^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha+1} \frac{\partial^m u(x,\tau)}{\partial \tau^m}\, d\tau, & \text{for } m-1 < \alpha < m \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha = m \in N \end{cases}$$

2. Analysis of the method

The basic definitions and fundamental operations of generalized differential transform method are defined in [55–57] as follows:

Definition 2.1. The generalized differential transform of the function $u(x,y)$ is given as follows:

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)} \left[\left(D_{x_0}^\alpha\right)^k \left(D_{y_0}^\beta\right)^h \right]_{(x_0,y_0)} \tag{2.1}$$

where $\left(D_{x_0}^\alpha\right)^k = D_{x_0}^\alpha \cdot D_{x_0}^\alpha \cdots D_{x_0}^\alpha$.

Definition 2.2. The generalized differential inverse transform of $U_{\alpha,\beta}(k,h)$ is defined as follows:

$$u(x,y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U_{\alpha,\beta}(k,h)(x-x_0)^{k\alpha}(y-y_0)^{h\beta} \tag{2.2}$$

The fundamental operations of generalized differential transform method are listed in Table 1: (see [55–57]).

3. Applications

In this section, we will investigate the solutions of two examples of the reaction–diffusion equations.

Example 1. Let us consider Eq. (1.2) with $f(u) = 6u(1-u)$, then we have the time-fractional Fisher equation

$$D_t^\alpha u = D_x^2 u + 6u(1-u), \qquad 0 < \alpha \leqslant 1, \quad x \in \Re, \quad t > 0 \tag{3.1}$$