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An elliptic averaging method for harmonically excited oscillators with a purely non-linear non-negative real-power restoring force

Zvonko Rakaric, Ivana Kovacic *

Department of Mechanics, Faculty of Technical Sciences, University of Novi Sad, 21000 Novi Sad, Serbia

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A B S T R A C T

Harmonically excited oscillators with a purely non-linear non-negative real-power restoring force are considered in this paper. The solution for motion is assumed in the form of a Jacobi cn elliptic function, the frequency and the parameter of which are obtained from the exact period of motion calculated from the energy conservation law. The parameter of the elliptic function is found to be negative for under-linear restoring forces and positive for over-linear restoring forces. By taking into account the time variation of the parameter of the elliptic function, a new elliptic averaging method is developed. The use of the equations derived is illustrated on the examples of oscillators with van der Pol damping and linear viscous damping with various integer and non-integer powers of the restoring force. New insights into dynamics of these oscillators are engendered. Numerical confirmations of analytical results are provided.

1. Introduction

In this paper harmonically excited non-linear oscillators are considered, governed by the following non-dimensional equation of motion

$$\frac{d^2\xi}{d\tau^2} + \varepsilon f\left(\xi, \frac{d\xi}{d\tau}\right) + \mathrm{sgn}(\xi)|\xi|^{\alpha} = \varepsilon \bar{F}\cos\Omega\tau, \tag{1}$$

where ξ is the non-dimensional displacement; τ is non-dimensional time; ε is a perturbation parameter; $F \equiv \varepsilon\bar{F}$ and Ω are the magnitude and frequency of harmonic excitation, respectively, with the former being of order ε; α is the power of the restoring force that can be any non-negative real number; f stands for any autonomous force that depends on the displacement and/or velocity.

A purely non-linear restoring force, which is a single-term, power-form function of the generalized coordinate, has recently attracted considerable attention, as it appears in different physical and engineering systems [1]. Conservative oscillators with this type of restoring force have been the subject of extensive research in the last decade (see, for example, [2] and the references citied therein as well as [3] for wider background), unlike non-conservative free and forced oscillators, which have not been studied so extensively. In [1], free conservative and non-conservative oscillations of purely non-linear autonomous oscillators are considered. To that end, elliptic functions are used to approximate motion for all non-negative real powers of the restoring force and, based on that, a new method of averaging is developed.

* Corresponding author. Tel.: +381 214852241; fax: +381 21450207.
E-mail address: ivanakov@uns.ac.rs (I. Kovacic).

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The use of elliptic functions and averaging for finding motion of non-linear systems have been proven to yield reliable and accurate results [4–16] as these functions contain higher-order harmonic components. Elliptic averaging has mainly been used to study free non-linear oscillations [4–12,14], while forced non-linear vibrations have been considered in this way by Roy [13] and Okabe et al. [15,16]. In Roy's method [13], the equation of motion is transformed into an autonomous system with respect to energy and the so-called resonant phase angle, but the solution for motion was found only for the cases when the exact solution of the unperturbed oscillator is known a priori. These cases include a mathematical pendulum (the restoring force is a Sine function of the displacement), a single-well Duffing oscillator (the restoring force consists of a positive linear and a positive cubic term) and a canonical escape oscillator (the restoring force has the form of a positive linear-negative quadratic term). In [15,16], elliptic averaging is proposed for the oscillators the restoring force of which contains a linear and a cubic term of different signs. This implies that there is no result obtained by elliptic averaging that enables one to study purely non-linear one-degree-of-freedom harmonically excited oscillators modeled by a general equation (1). The method presented in this paper is developed to fill the detected gap and represents a continuation of the study of free oscillators given in [1].

2. Novel elliptic method

2.1. Motion of conservative oscillators

The solution of conservative oscillators needs to be determined first, as it will serve as a generating solution for the motion of the forced oscillators (1). According to the assumptions introduced, both the autonomous term $f(\xi, d\xi/d\tau)$ and the harmonic force represent perturbations of the conservative oscillators, whose equation of motion is derived from Eq. (1) by putting $\varepsilon = 0$:

$$\frac{d^2\xi}{d\tau^2} + \mathrm{sgn}(\xi)|\xi|^{\alpha} = 0. \tag{2}$$

Initial conditions are prescribed as $\xi(0) = a$, $d\xi/d\tau(0) = 0$, where a is a real constant.

In order to encounter for the non-linearity, i.e. the fact that the power α can take any non-negative real value, the approximate solution for motion and the velocity of Eq. (2) are assumed in the form of the Jacobi elliptic functions instead of usual trigonometric functions. The motivation stems from the fact that this elliptic function contains higher harmonics, which are, in general, of great importance for non-linear systems. Thus, the solutions for the displacement and the velocity have the form analogous to that for linear systems:

$$\xi(\tau) = a\mathrm{cn}[\psi, m], \quad \frac{d\xi}{d\tau} = -a\omega\ \mathrm{sn}[\psi, m]\mathrm{dn}[\psi, m], \tag{3a, b}$$

where a is a constant amplitude of motion defined by the initial displacement, ψ is the complete phase and the parameter m is the square of the modulus [17]. The complete phase ψ is defined by

$$\psi(\tau) = \omega\tau + \varphi, \tag{4}$$

where ω and φ are the frequency and the phase of the response, respectively.

To obtain the frequency ω and the square of the modulus m, the energy integral of the oscillators (2) is considered

$$\frac{1}{2}\left(\frac{d\xi}{d\tau}\right)^2 + \frac{|\xi|^{\alpha+1}}{\alpha+1} = \frac{|a|^{\alpha+1}}{\alpha+1}. \tag{5}$$

It should be emphasized that by a proper non-dimensionalization, the value of the amplitude a can always be made equal to unity (see [1] for details), which will be used later on for simplification.

By using Eq. (5), the following expression for the exact period of oscillations can be derived [1]

$$T = 4\int_0^a \frac{d\xi}{|\frac{d\xi}{d\tau}|} = 4\sqrt{\frac{\alpha+1}{2}}\int_0^a \frac{d\xi}{\sqrt{|a|^{\alpha+1} - |\xi|^{\alpha+1}}} = T_{ex}^{\mathrm{ND}} a^{\frac{1-\alpha}{2}}, \tag{6}$$

with T_{ex}^{ND} being

$$T_{ex}^{\mathrm{ND}} = 4\sqrt{\frac{\alpha+1}{2}}\frac{\sqrt{\pi}\Gamma\left[1 + \frac{1}{1+\alpha}\right]}{\Gamma\left[\frac{1}{2} + \frac{1}{1+\alpha}\right]}, \tag{7}$$

where Γ represents the Euler Gamma function [17]. When the initial amplitude is equal to unity, the exact period T becomes equal to the one described by Eq. (7), which is plotted in Fig. 1 for illustration. For the linear case $\alpha = 1$, this expression gives the well-known result 2π. In under-linear cases ($\alpha < 1$), this period is smaller than this value. When $\alpha > 1$ (over-linear case), this period is higher than 2π and increases with the power α.