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Travelling wave solutions for the nonlinear dispersion Drinfel'd–Sokolov ($D(m,n)$) system

Xijun Deng [a,*], Jinlong Cao [b], Xi Li [c]

[a] School of Information and Mathematics, Yangtze University, Jingzhou, Hubei 434023, China
[b] Department of Mathematics, Shanghai jiaotong University, Shanghai 200240, China
[c] Department of Mathematics, University of Kansas, KS 66045, USA

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ABSTRACT

In this paper, travelling wave solutions for the nonlinear dispersion Drinfel'd–Sokolov system (called $D(m,n)$ system) are studied by using the Weierstrass elliptic function method. As a result, more new exact travelling wave solutions to the $D(m,n)$ system are obtained including not only all the known solutions found by Xie and Yan but also other more general solutions for different parameters m,n. Moreover, it is also shown that the $D(m,1)$ system with linear dispersion possess compacton and solitary pattern solutions. Besides that, it should be pointed out that the approach is direct and easily carried out without the aid of mathematical software if compared with other traditional methods. We believe that the method can be widely applied to other similar types of nonlinear partial differential equations (PDEs) or systems in mathematical physics.

1. Introduction

It is well known that investigating the exact travelling wave solutions to nonlinear evolution equations play an important role in the study of nonlinear physical phenomena. In order to obtain the exact solutions, a number of methods have been proposed, such as the homogeneous balance method [1], the hyperbolic function expansion method [2], Jacobi elliptic function method [3] and F-expansion method [4], homotopy analysis method [5,6], the bifurcation theory method of dynamical systems [7,8], Weierstrass elliptic function method [9]. Among these methods, Weierstrass elliptic function method is a powerful mathematic tool to solve nonlinear evolution equations. By using this method, many kinds of important nonlinear evolution equations have been solved successfully [10,11].

The usual Drinfel'd–Sokolov system reads

$$\begin{cases} u_t + (v^2)_x = 0, \\ v_t + a v_{xxx} + b u_x v + c u v_x = 0. \end{cases} \tag{1}$$

where a,b,c are constants, this system is regarded as an example of a system of nonlinear equations possessing Lax pairs of a special form [12]. In [13], Wang gave its recursion, Hamiltonian, symplectric and cosymplectric operators and roots of its symmetries and scaling symmetry.

* Corresponding author. Tel.: +86 1532 7692907.
E-mail address: xijundeng@yahoo.com.cn (X. Deng).

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The generalized Drinfel'd–Sokolov system reads

$$\begin{cases} u_t + (v^m)_x = 0, \\ v_t + av_{xxx} + bu_x v + cuv_x = 0. \end{cases} \tag{2}$$

By using the tanh method and the sine–cosine method, Wazwaz [14] obtained some exact travelling wave solutions with compact and noncompact structures of Eq. (2).

In this paper, we shall consider the travelling wave solutions of the following nonlinear dispersion Drinfel'd–Sokolov (simply called $D(m,n)$) system

$$\begin{cases} u_t + (v^m)_x = 0, \\ v_t + a(v^n)_{xxx} + bu_x v + cuv_x = 0. \end{cases} \tag{3}$$

By using some transformations, Xie and Yan [15] obtained some types of exact travelling wave solutions to Eq. (3), which include compactons, solitons, solitary patterns and periodic solutions.

The objective of this paper is to further investigate the travelling wave solutions of $D(m,n)$ Eq. (3) systematically, by applying the Weierstrass elliptic function method. As a consequence, a new set of exact travelling wave solutions has been obtained, which is more comprehensive and includes all the results described in [15] as special cases. Similar to Xie and Yan's results [15], our results also show that the $D(m,1)$ system with linear dispersion possess compacton and solitary pattern solutions.

The rest of this paper is organized as follows. In Section 2, we first outline the Weierstrass elliptic function method which will be used in the next section. In Section 3, we give some general and particular travelling wave solutions of Eq. (3). Finally, some conclusions are given in Section 4.

2. Weierstrass elliptic functions

Let us consider the following nonlinear differential equation

$$\left(\frac{d\phi}{dt}\right)^2 = a_0\phi^4 + 4a_1\phi^3 + 6a_2\phi^2 + 4a_3\phi + a_4 \equiv f(\phi), \tag{4}$$

As is well-known [16,17] that the solutions $\phi(t)$ of (4) can be expressed in terms of elliptic functions $\wp$. It reads as

$$\phi = \phi_0 + \frac{1}{4}f'(\phi_0)\left(\wp(t;g_2,g_3) - \frac{1}{24}f''(\phi_0)\right)^{-1}, \tag{5}$$

where the primes denote differentiation with respect to ϕ and ϕ_0 is a simple root of $f(\phi)$.

The invariants g_2, g_3 of elliptic functions $\wp(t;g_2,g_3)$ are related to the coefficients of $f(\phi)$ by [18]

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2, \tag{6}$$

$$g_3 = a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4, \tag{7}$$

When g_2 and g_3 are real and the discriminant

$$\Delta = g_2^3 - 27g_3^2 \tag{8}$$

is positive, negative or zero, we have different behavior of $\wp(t)$. The conditions [9]

$$\Delta \neq 0 \text{ or } \Delta = 0, \quad g_2 > 0, \quad g_3 > 0, \tag{9}$$

lead to periodic solutions, whereas the conditions

$$\Delta = 0, \quad g_2 \geqslant 0, \quad g_3 \leqslant 0, \tag{10}$$

lead to solitary wave solutions.

If $\Delta = 0$, then $\wp(t;g_2,g_3)$ degenerates into trigonometric or hyperbolic functions [19]. Thus, periodic solutions according to Eq. (5) are determined by

$$\phi(t) = \phi_0 + \frac{f'(\phi_0)}{4\left[-\frac{e_1}{2} - \frac{f''(\phi_0)}{24} + \frac{3}{2}e_1\csc^2\left(\sqrt{\frac{3}{2}e_1}t\right)\right]}, \quad \Delta = 0, \quad g_3 > 0, \tag{11}$$

and solitary wave solutions by

$$\phi(t) = \phi_0 + \frac{f'(\phi_0)}{4[e_1 - \frac{f''(\phi_0)}{24} + 3e_1 csch^2(\sqrt{3e_1}t)]}, \quad \Delta = 0, \quad g_3 < 0, \tag{12}$$

where $e_1 = \sqrt[3]{|g_3|}$ in Eq. (11) and $e_1 = \frac{1}{2}\sqrt[3]{|g_3|}$ in Eq. (12).