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Bifurcations of travelling wave solutions for a class of nonlinear fourth order variant of a generalized Camassa–Holm equation

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ABSTRACT

In this paper, by using the bifurcation theory of dynamical systems for a class of nonlinear fourth order variant of a generalized Camassa–Holm equation, the existence of solitary wave solutions, breaking bounded wave solutions, compacton solutions and non-smooth periodic wave solutions are obtained. Under different parametric conditions, various sufficient conditions to guarantee the existence of the above solutions are given. Some exact explicit parametric representations of the above waves are determined.

1. Introduction

In our recent paper, by using sine–cosine method, we investigated the nonlinear dispersive variants CH($-n$, $-n$, $-m$) of the generalized Camassa–Holm equation [1] (simply called GCH($-n$, $-n$, $-m$))

$$u_{tt} = (au + bu^{-n} + du^{-m})_{xx} + k(u^{-n})_{xxtt}, \tag{1.1}$$

where $a,\ k > 0,\ bd \neq 0,\ m > n \geqslant 1$. It is shown that this class gives compactons, conventional solitons, solitary patterns and periodic solutions. It is also found that the qualitative change in the physical structure of solutions depends mainly on the exponent of the wave function $u(x,t)$, positive or negative, and on the coefficient of $(u^{-n})''$ as well. It is very important to consider the dynamical bifurcation behavior for the travelling wave solutions of (1.1). There are some interesting problems: Does an exact travelling wave solution obtained by the computer algebraic method really satisfy the given travelling equation? What is the dynamical behavior of the known exact travelling wave solutions? How do the travelling wave solutions depend on the parameters of the system? Are there the dynamics of the so-called compacton solutions for (1.1)? In this paper, we shall study all travelling wave solutions in the parameter space of this system. Let $u(x,t) = \phi(x - ct) = \phi(\xi)$, where c is the wave speed. Then (1.1) becomes to

$$c^2\phi'' = (a\phi + b\phi^{-n} + d\phi^{-m})'' + c^2k(\phi^{-n})^{(4)}, \tag{1.2}$$

where "′" is the derivative with respect to ξ. Integrating (1.2) twice and setting two integration constants as zero, we have

$$q\phi^{m+1} + p\phi^{m-n} + 1 + r[n(n+1)\phi^{m-n-2}(\phi')^2 - n\phi^{m-n-1}\phi''] = 0, \tag{1.3}$$

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where $q = \frac{a-c^2}{d},\ p = \frac{b}{d},\ r = \frac{c^2 k}{d}$. Eq. (1.3) is equivalent to the two-dimensional systems as follows

$$\frac{d\phi}{d\xi} = y,\ \frac{dy}{d\xi} = \frac{1 + p\phi^{m-n} + q\phi^{m+1} + rn(n+1)\phi^{m-n-2}y^2}{rn\phi^{m-n-1}} \tag{1.4}$$

with the first integral for $n > 1$

$$H(\phi, y) = \frac{rn}{2\phi^{2(n+1)}}y^2 + \left[\frac{q}{n-1}\phi^{-n+1} + \frac{p}{2n}\phi^{-2n} + \frac{1}{(m+n)}\phi^{-m-n}\right] = h \tag{1.5}$$

and with the first integral for $n = 1$

$$H_1(\phi, y) = \frac{r}{2\phi^4}y^2 - \left[q\ln\phi - \frac{p}{2}\phi^{-2} - \frac{1}{(m+1)}\phi^{-m-1}\right] = h. \tag{1.6}$$

System (1.4) is a 5-parameter planar dynamical system depending on the parameter group (m, n, p, q, r). For different m, n and a fixed r, we shall investigate the bifurcations of phase portraits of (1.4) in the phase plane (ϕ, y) as the parameters p, q are changed. Here we are considering a physical model where only bounded travelling waves are meaningful. So we only pay attention to the bounded solutions of (1.4).

Suppose that $\phi(\xi)$ is a continuous solution of (1.4) for $\xi \in (-\infty, \infty)$ and $\lim_{\xi\to\infty}\phi(\xi) = \alpha, \lim_{\xi\to-\infty}\phi(\xi) = \beta$. Recall that (i) $\phi(x,t)$ is called a solitary wave solution if $\alpha = \beta$; (ii) $\phi(x,t)$ is called a kink or anti-kink solution if $\alpha \neq \beta$. Usually, a solitary wave solution of (1.1) corresponds to a homoclinic orbit of (1.4); a kink (or anti-kink) wave solution (1.1) corresponds to a heteroclinic orbit (or the so-called connecting orbit) of (1.4). Similarly, a periodic orbit of (1.4) corresponds to a periodically travelling wave solution of (1.1). Thus, to investigate all possible bifurcations of solitary waves and periodic waves of (1.1), we need to find all periodic annuli and homoclinic orbits of (1.4), which depend on the system parameters. The bifurcation theory of dynamical systems (see [2,3]) plays an important role in our study.

We notice that the right hand of the second equation in (1.4) is not continuous when $\phi = 0$. In other words, on the above straight line of the phase plane (ϕ, y), ϕ''_ξ has no definition. It implies that the smooth system (1.1) sometimes has non-smooth travelling wave solutions. This phenomenon has been studied by some authors (see [4–10]). We claim that the existence of a singular straight line for a travelling wave equation is the original reason why travelling waves lose their smoothness.

The paper is organized as follows. In Section 2, we discuss bifurcations of phase portraits of (1.4), where explicit parametric conditions will be derived. In Section 3, some explicit parametric representations of the bounded travelling wave solutions are given. In Section 4, the existence of smooth solitary wave solutions, kink or anti-kink wave solutions, compacton solutions and non-smooth periodic wave solutions of (1.4) are discussed.

2. Bifurcations of phase portraits of (1.4)

In this section, we study all possible periodic annuluses defined by the vector fields of (1.4) when the parameters p, q are varied.

Let $d\xi = rn\phi^{m-n-1}d\zeta$. Then, except on the straight lines $\phi = 0$, the system (1.4) has the same topological phase portraits as the following system

$$\frac{d\phi}{d\zeta} = rn\phi^{m-n-1}y, \quad \frac{dy}{d\zeta} = 1 + p\phi^{m-n} + q\phi^{m+1} + rn(n+1)\phi^{m-n-2}y^2. \tag{2.1}$$

Now, the straight lines $\phi = 0$ is an integral invariant straight line of (2.1).

Denote that

$$f(\phi) = 1 + p\phi^{m-n} + q\phi^{m+1}, \quad f'(\phi) = \phi^{m-n-1}[p(m-n) + q(m+1)\phi^{n+1}]. \tag{2.2}$$

For $n = 2l, m = 2m_1, m_1 > l, l, m_1 \in Z^+$, when $\phi = \phi_0 = \left[-\frac{p(m-n)}{q(m+1)}\right]^{\frac{1}{n+1}}$, $f'(\phi_0) = 0$. We have $f(\phi_0) = 1 + p\left[-\frac{p(m-n)}{q(m+1)}\right]^{\frac{m-n}{n+1}} + q\left[-\frac{p(m-n)}{q(m+1)}\right]^{\frac{m+1}{n+1}}$, which imply the relations in the (p, q)-parameter plane

$$L_a: \quad q = \frac{m-n}{m+1}(-p)^{\frac{m+1}{m-n}}\left(\frac{n+1}{m+1}\right)^{\frac{n+1}{m-n}}, \quad p < 0,\ q > 0.$$

$$L_b: \quad q = -\frac{m-n}{m+1}(-p)^{\frac{m+1}{m-n}}\left(\frac{2m-n+1}{m+1}\right)^{\frac{n+1}{m-n}}, \quad p < 0, q < 0.$$

For $n = 2l, m = 2m_1 + 1, m_1 \geqslant l, l, m_1 \in Z^+$, when $\phi = \phi_0 = \left[-\frac{p(m-n)}{q(m+1)}\right]^{\frac{1}{n+1}}$, $f'(\phi_0) = 0$. We have $f(\phi_0) = 1 + p\left[-\frac{p(m-n)}{q(m+1)}\right]^{\frac{m-n}{n+1}} + q\left[-\frac{p(m-n)}{q(m+1)}\right]^{\frac{m+1}{n+1}}$, which imply the relations in the $(p, q)-$parameter plane