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On the solutions of a second order nonlinear system with almost periodic forcing ☆

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ABSTRACT

A second order almost periodic perturbed nonlinear system with small parameter is discussed in this paper. Based on some analytical techniques and the method of averaging, some sufficient conditions are obtained for existence of one almost periodic solution. Also some suitable conditions are given for existence of two almost periodic solutions. The obtained new results generalized the known results in Seifert [1,3] and He [2,4]. Finally, some applications are presented to illustrate that our results are a good generalization of the known results.

1. Introduction

In real world phenomenon, the environment varies due to the factors such as seasonal effects of weather, food supplies, mating habits, harvesting. So it is usual to assume the periodicity of parameters in the systems. However, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. For this reason, the assumption of almost periodicity is more realistic, more important and more general when we consider the effects of the environmental factors. We say that a continuous function $f(t)$ is an almost periodic function on R if for all $\varepsilon > 0$, there exists $\tau = \tau(\varepsilon)$ such that for all $t \in R$, $\|f(t+\tau - f(\tau)\| < \varepsilon$. The number τ is called a ε-translation number of $f(t)$. For the concept of almost periodicity and its significance, the reader can refer to [1–19] and the reference cited therein.

One of the important issue of the almost periodic differential equations is to study the almost periodic solution of the forced perturbation systems. For this reason, Seifert [1] considered the following perturbed differential system:

$$\frac{dx}{dt} = [A + \varepsilon C(t)]x + \varepsilon g(x, \varepsilon) + \varepsilon b(t), \tag{1}$$

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and

$$\frac{d^2x}{dt^2} + (1-\varepsilon v) + \varepsilon g(x,\varepsilon)\frac{dx}{dt} + \varepsilon x^3 + b(t),$$

where $x, g(x,\varepsilon)$ and $b(t)$ belong to R^n, $\varepsilon \in [0,\varepsilon_0]$, it is a small parameter. Under some suitable conditions and by the method of averaging, Seifert [1] obtained some sufficient conditions for the existence of almost periodic solution of system (1).

Considering the physical significance, He [2] considered a perturbed differential system

$$\begin{cases} \frac{dx_1}{dt} = x_2, \\ \frac{dx_2}{dt} = -a^2x_1 + \varepsilon(v - \mu\cos t)x_1 + \varepsilon^{2k+1}h(x_1) + f(t). \end{cases} \tag{2}$$

Taking $a = \lambda_j = 1,\ \mu = 0,\ h(x_1) = x_1^3$, system (2) reduces to the system which was studied by Seifert [1]. Taking $a = \lambda_j = 1,\ \mu = 0,\ h(x_1) = \sum_{j=1}^{2k+1} a_j x_1^j$, system (2) reduces to the system which was well studied by Seifert [3].

For convenience, the following assumptions are valid throughout this paper.

(H_1) A is an $n\times n$ matrix which is similar to the diagonal matrix with pure imaginary entries, $C(t)$ and $b(t)$ are almost periodic, $g(x,\varepsilon)$ and $\frac{\partial g(x,\varepsilon)}{\partial x}$ are continuous in $R^n \times [0,\varepsilon_0]$.

(H_2) $h(x_1) = \sum_{j=1}^{2k+1} a_j x_1^j,\ \ v a_{2k+1} < 0$, k is a positive integer, Fourier series of real function $f(t)$ is

$$f(t) \approx \sum_{j=1}^{\infty}(A_j\cos\lambda_j t + B_j \sin\lambda_j t), \quad A_j^2 + B_j^2 > 0.$$

By the method of averaging, He [4], Hale [5] and Jiang [6] studied nonlinear system with small parameter $\frac{dx}{dt} = \varepsilon f(t,x,\varepsilon)$. For more recent works about almost periodic solutions for the perturbation systems, we also refer to Ruan [7], Lin [8], He [10], Xia et al [11], Xia et al [13], Schmitt and Ward [18], Berger and Chen [19] and the references cited therein.

In this paper we study a more generalized nonlinear differential equation

$$\begin{cases} \frac{dx_1}{dt} = x_2 - \varepsilon(v_2 + \mu_2\cos t)x_2 - \varepsilon^{2k+1}h(x_2), \\ \frac{dx_2}{dt} = -a^2x_1 + \varepsilon(v_1 - \mu_1\cos t)x_1 + \varepsilon^{2k+1}h(x_1) + f(t). \end{cases} \tag{3}$$

Based on employing new analytical techniques and the method of averaging, some sufficient conditions are obtained for the existence of one and two almost periodic solutions of nonlinear system (3), respectively. These new results generalize the results in Seifert [1,3] and He [2,4].

2. Main results and their proofs

In order to prove our main results, we need the following Definition 1 and Lemmas 1 and 2.

Definition 1. Let $f_0(x) = m(f(t,x,0)) = \lim_{T\to+\infty}\frac{1}{T}\int_0^T f(t,x,0)dt$, then $f_0(x)$ is called averaging of function $f(t,x,0)$ (see [3]).

Lemma 1. *Suppose that system (1) satisfies the following conditions:*

(1) $A, C(t), g(x,\varepsilon)$ and $b(t)$ satisfy condition (H_1);
(2) $C_0 = m(e^{-tA}C(t)e^{tA})$;
(3) $g_0(y) = m(e^{-tA}g(e^{tA}y,0))$;
(4) $b_0 = m(e^{-tA}b(t))$;
(5) equation

$$C_0y + g_0(y) + b_0 = 0 \tag{4}$$

has a solution $\bar{y}$ such that all eigenvalues of the matrix $C_0 + \frac{\partial g_0(\bar{y})}{\partial y}$ have nonzero real parts.

Then there exists an $\varepsilon^0 > 0$ such that for $0 \leqslant \varepsilon \leqslant \varepsilon^0$, system (1) has an almost periodic solution $x(t,\varepsilon)$ and $\lim_{\varepsilon\to 0}x(t,\varepsilon) = e^{tA}\bar{y}$ uniformly for t on R (see [1]).

Lemma 2. *Let $a = \lambda_j = 1,\ \mu = 0,\ h(x_1) = \sum_{j=1}^{2k+1} a_j x_1^j$ in system (2). Moreover, suppose that system (2) satisfies condition (H_2) and there exists a constant $v_0 = v_0(A_j, B_j, a_{2k+1}) > 0$, then for each v with $|v| > v_0$, there is an $\varepsilon^0 = \varepsilon^0(v)$, such that for $0 \leqslant \varepsilon \leqslant \varepsilon^0$, system (2) has an almost periodic solution $x(t,\varepsilon)$, and $\lim_{\varepsilon\to 0}\varepsilon x(t,\varepsilon) = (a\sin(t+\delta), a\cos(t+\delta))^T$ uniformly for t on R, where a and δ are two constants (see [3]).*

Now we shall state our main results.

In this paper we always assume that the condition (H_2) hold.

Theorem 1. *Let $a = \frac{1}{2},\ |\lambda_j| \neq \frac{1}{2},\ j = 1,2,\ldots, h(x_1) = \sum_{j=1}^{2k+1} a_j x_1^j$, $(\mu_1 + \frac{1}{4}\mu_2)^2 \neq 4(v_1 + \frac{1}{4}v_2)^2$ and $a_{2k+1}(v_1 + \frac{1}{4}v_2) < 0$, then there exists an $\varepsilon^0 > 0$ such that for $0 \leqslant \varepsilon \leqslant \varepsilon^0$, system (3) has an almost periodic solution $x(t,\varepsilon)$, if $(\mu_1 + \frac{1}{4}\mu_2)^2 > 4(v_1 + \frac{1}{4}v_2)^2$, and*