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Existence and uniqueness results for fractional integrodifferential equations with boundary value conditions

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ABSTRACT

In this paper, we study existence and uniqueness of fractional integrodifferential equations with boundary value conditions. A new generalized singular type Gronwall inequality is given to obtain an important a priori bounds. Existence and uniqueness results of solutions are established by virtue of fractional calculus and fixed point method under some weak conditions. An example is given to illustrate the results.

1. Introduction

The initial and boundary value problems for nonlinear fractional differential equations arise from the study of models of viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. [8,13,17,25,26]. During last years, the study of such kind of problems have received much attention from both theoretical and applied point of view. We will mention the following recent works on this topic [1–6,9,12,14–23], without try to be exhaustive. It is often that the authors use the following conditions: for the existence of solutions, the nonlinear term f must satisfy that, there exist functions $p, r \in C([0,1],[0,\infty))$ such that for $1 \geqslant t \geqslant 0$ and each $u \in R$,

$$|f(t,u)| \leqslant p(t)|u| + r(t),$$

while for the uniqueness, they consider that the nonlinear term f satisfy that, there exist functions $p, r \in C([0,1],[0,\infty))$ such that for each $1 \geqslant t \geqslant 0$ and any $u, v \in R$,

$$|f(t,u) - f(t,v)| \leqslant p(t)|u - v|.$$

In particular, Agarwal et al. [1] established sufficient conditions for the existence and uniqueness of solutions for various classes of initial and boundary value problem for fractional differential equations and inclusions involving the Caputo fractional derivative in finite dimensional spaces. Recently, some fractional differential equations and optimal controls in Banach spaces were studied by Balachandran and Park [5], El-Borai [9], Henderson and Ouahab [10], Hernandez et al. [11], Mophou and N'Guérékata [18] and Wang et al. [22,24].

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The existence of solutions for this kind of BVP has been studied by Benchohra et al. [7]. Let us mention, however, the assumptions on f are strong (f is continuous and satisfies uniformly Lipschitz condition or uniformly bounded). We will present the new existence and uniqueness results for the fractional BVP (1.1) by virtue of fractional calculus and fixed point method under some weak conditions. Compared with the results appeared in [7], there are at least two differences: (i) the assumptions on f are more general and easy to check; (ii) a priori bounds is established by a new singular type Gronwall inequality (Lemma 3.1) given by us.

In this paper, we consider the existence of solutions of the following boundary value problems:

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t), (Sy)(t)), \quad 0 < \alpha < 1, \quad t \in J = [0, T], \\ ay(0) + by(T) = c, \end{cases} \tag{1.1}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order $\alpha, f : J \times X \times X \to X$ where X is a Banach spaces and a, b, c are real constants with $a + b \neq 0$, and S is a nonlinear integral operator given by

$$(Sy)(t) = \int_0^t k(t,s)y(s)ds,$$

with $\gamma_0 = \max\{\int_0^t k(t,s)ds : (t,s) \in J \times J\}$ where $k \in C(J \times J, \mathbb{R}^+)$.

This paper is organized as follows. In Section 2 we give the notations and some concepts and preparation results. A generalized singular Gronwall type inequality is used to proof the main results in Section 3. The first one of they is based on the application of Banach contraction principle, and the second result is based on Schaefer's fixed point theorem. In the last section we apply the main results to certain particular case.

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary results, which will be used throughout this paper. We denote $C(J, X)$ the Banach space of all continuous functions from J into X with the norm $\|y\|_\infty := \sup\{\|y(t)\| : t \in J\}$. For measurable functions $m : J \to \mathbb{R}$, define the norm $\|m\|_{L^p(J,\mathbb{R})} = \left(\int_J |m(t)|^p dt\right)^{\frac{1}{p}}, \ 1 \leqslant p < \infty$. We denote $L^p(J, \mathbb{R})$ the Banach space of all Lebesgue measurable functions m with $\|m\|_{L^p(J,\mathbb{R})} < \infty$.

We will need the following basic definitions and properties of the fractional calculus theory to follow the contents of this paper. For more details, see, for example, [13].

Definition 2.1. The Riemann–Liouville fractional integral of order $\alpha \in \mathbb{R}_+$, that is a positive real number, of a function suitable function h, for example $h \in L^1([a, b], \mathbb{R})$, is defined by

$$I^\alpha_{a+}h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where $a \in \mathbb{R}$ and Γ is the Gamma function.

Here we will consider $n = -[-\alpha]$, where $[\cdot]$ denotes the integer part of the argument, and $\alpha > 0$.

Definition 2.2. For a suitable function h given on the interval $[a, b]$, the Riemann–Liouville fractional derivative of order α of h, is defined by

$$(D^\alpha_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds.$$

Definition 2.3. For a suitable function h given on the interval $[a, b]$, the Caputo fractional order derivative of order α of h, is defined by

$$({}^cD^\alpha_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds.$$

Lemma 2.1. *Let $\alpha > 0$, then the differential equation ${}^cD^\alpha h(t) = 0$ has the following general solution*

$$h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}, \ i = 0, 1, 2, \ldots, n-1, n$, with $n = -[-\alpha]$.

Lemma 2.2. *Let $\alpha > 0$, then*

$$I^\alpha({}^cD^\alpha h)(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$