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Letter to the Editor

Periodic wave solutions of coupled integrable dispersionless equations by residue harmonic balance

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ABSTRACT

We introduce the residue harmonic balance method to generate periodic solutions for nonlinear evolution equations. A PDE is firstly transformed into an associated ODE by a wave transformation. The higher-order approximations to the angular frequency and periodic solution of the ODE are obtained analytically. To improve the accuracy of approximate solutions, the unbalanced residues appearing in harmonic balance procedure are iteratively considered by introducing an order parameter to keep track of the various orders of approximations and by solving linear equations. Finally, the periodic solutions of PDEs result. The proposed method has the advantage that the periodic solutions are represented by Fourier functions rather than the sophisticated implicit functions as appearing in most methods.

1. Introduction

There exists various methods in literature dealing with the approximate solutions to nonlinear evolution equations, for example, the inverse scattering method [1], the Hirota method [2,3], Bäcklund and Darboux transformation [4–7], Wronskian technique [8,9],homogeneous method [10,11], tanh method [12], Jacobi elliptic method [13], sine–cosine method [14,15], non-linearization method [16,17], homotopy perturbation method [18,19],and Adomian Pade approximation [20,21], etc. These methods are good for shock wave and solitary wave solutions of nonlinear equations.

A transformed rational function method, proposed by Ma and Lee [22], is used to obtain exact solutions to the 3 + 1 dimensional Jimbo–Miwa equation. This new method provides a more systematical and convenient handling of the solution procedure of nonlinear equations, unifying the tanh function type methods, the homogeneous balance method, the exp-function method, the mapping method, and the F-expansion type methods. Its key point is to search for rational solutions to associated ordinary differential equations transformed from given partial differential equations. However, to get the N-soliton and N-wave solution of the PDE, we may consider the linear superposition principle [23] and multiple exp-function method [24], the latter of which is the most general based on Fourier theory. They have been used to obtain many N-wave solutions of the (3 + 1)-dimensional potential-Yu–Toda–Sasa–Fukuyama equation, the (3 + 1)-dimensional KP equations etc.

Due to the availability of computer symbolic systems which allow us to perform some complicated and tedious algebraic calculation, searching for accurate solutions of nonlinear PDEs is still one of the most exciting and active research area.

In general, the periodic wave solutions will be helpful in the theoretical and numerical study of the nonlinear evolution systems. In the study of equations describing wave phenomena, one of the fundamental objects is the traveling wave solution [24] that a solution possesses constant form moving with a fixed velocity and changeless shapes during propagation. Wherein, the wave transform $\xi = x - ct$, *is* always used to convert a nonlinear PDE to nonlinear ODEs. However, solving

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the nonlinear ODEs is also a very difficult target to achieve. Typical method by which the solution theory is used is perturbation method [25] for nonlinear ODEs by a large number of researchers. But almost all perturbation methods are based on small parameters so that the approximate solutions are expanded in series of small parameters. The smallness of the small parameter determines not only the accuracy of the approximations but also the validity of perturbation method. We do not use perturbation and do not assume small parameters; rather, we use an order parameter to keep track of the order of approximations.

In this paper, a new method for approximations to the time–space periodic solutions of nonlinear evolution equations, i.e., the residue harmonic balance method [26,27] is introduced. The primary idea of this method is that an order parameter [28] is introduced into the residue of the Fourier truncation series in the harmonic equations. The unbalanced residues due to Fourier truncation are considered iteratively by solving linear algebraic equations to improve the accuracy of the solutions successively. The most interesting features of the proposed method are its simplicity and its excellent accuracy in a wide range of parameter values of the system. The proposed method is first outlined and the three coupled integrable dispersionless equations are transformed into the canonical Duffing form for subsequent analysis.

2. The proposed method

2.1. Travelling wave reduction method

Consider a nonlinear evolution equation

$$P(x,t,u,u_t,u_x,u_{xx},u_{xt},u_{tt},u_{xxx}\ldots)=0, \tag{1}$$

where $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, ... and u is the unknown function of the independent coordinates x and t. Let

$$u(x,t)=v(\xi),\xi=x-ct, \tag{2}$$

where c is a nonzero unknown constant representing the speed of the propagating waves. Substituting Eq. (2) into Eq. (1) yields an ordinary differential equation of $v(\xi)$

$$O(\xi,v,\dot{v},\ddot{v},\ldots;\lambda)=0, \tag{3}$$

where the over dot means derivatives of v with respect to ξ, and λ denotes the parameters. For convenience, some of the constants of integration may be taken to be zero.

2.2. Residue harmonic balance method

Most of PDEs may be transformed to the following second order ODE whose periodic solution with unknown period is to be found

$$\ddot{v}+f(v)=0,\ v(0)=A,\ \dot{v}(0)=0. \tag{4}$$

Now, by introducing a new time variable $\tau = \omega\xi$, the nonlinear system (4) becomes

$$\omega^2 v''+f(v)=0,\ v(0)=A,\ v'(0)=0. \tag{5}$$

Here the prime denotes differentiation of v with respect to τ. The new independent variable is chosen in such a way that the solution to Eq. (5) is a periodic function of τ having period 2π. Both the periodic solution $v(\tau)$ and frequency ω depend on A. We consider a special case that $f(v)$ is an odd function of u i.e., $f(-v) = -f(v)$. Therefore, a general initial approximation of Eq. (5) can be defined by

$$v_0(\tau)=A\cos(\tau). \tag{6}$$

Now, we introduce an order parameter $p \in [0,1]$ to keep track of the various order of approximation and rewrite $v(\tau)$ as $v(\tau,p)$ and ω as $\omega(p)$,

$$\begin{cases}\omega(p)=\omega_0+p\omega_1+p^2\omega_2+\cdots,\\ v(\tau,p)=v_0(\tau)+pv_1(\tau)+p^2v_2(\tau)+\cdots\end{cases} \tag{7}$$

The kth order correction $v_k(\tau)$ is expanded by the Fourier series, i.e.,

$$v_k(\tau)=\sum_{i=1}^{k}a_{k,i}\{\cos(\tau)-\cos[(2i+1)\tau]\},\quad k=1,2,\ldots \tag{8}$$

Then the final solution will be given by

$$\begin{cases}\omega=\omega_0+\omega_1+\omega_2+\cdots,\\ v(\tau)=v_0(\tau)+v_1(\tau)+v_2(\tau)+\cdots\end{cases}$$

Substituting Eq. (7) into Eq. (5) leads to