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Short communication

Note on unsteady viscous flow on the outside of an expanding or contracting cylinder

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ABSTRACT

In this paper, the viscous flow on the outside of an expanding or contracting cylinder is studied. The governing Navier–Stokes equations are transformed into a similarity equation, which is solved by a shooting method. The solution is an exact solution to the unsteady Navier–Stokes equations. Results show both trivial and non-trivial solutions. For trivial solutions, there is no axial flow induced during the cylinder expansion or contraction. However, for the non-trivial solutions which only exist for cylinder expansion, an axial flow is generated and its strength increases with the increase in expansion speed.

1. Introduction

Exact solutions of the Navier–Stokes (NS) equations play important roles in the development of fluid mechanics and provide interesting physical insights. Meanwhile, they are used as benchmarks for numerical code validation. Wang summarized the available exact solutions of the unsteady [1] and of the steady state NS equations [2]. As pointed out by Wang, "On the other hand, similarity solutions, where v is implicit in the similarity transforms, and where universal curves can be obtained once and for all, are exact solutions". However, exact solutions do not necessarily mean the solutions are in analytical closed form. For example, the steady stagnation point flow and the von Karman rotating disk problem were both obtained in similarity form and solved numerically. Both are classical examples of exact solutions to the NS equations [2]. The flow inside a channel or a tube with a stretching wall was solved by Brady and Acrivos [3]. The flow outside a stretching tube with acceleration was analyzed by Wang [4]. All these solutions are exact solutions of the whole NS equations. The unsteady flow inside a tube with time dependent diameter was first studied by Uchida and Aoki [5] and by Skalak and Wang [6], and they calculated the internal flow velocity and pressure due to tube expansion or contraction. Later the unsteady flow in a tube with both axial and radial motion was studied [7–9]. Recently, the flow was extended to a tube with porous walls by considering mass suction and injection [10]. But the flow at the outside of the tube or cylinder did not receive much attention in the literature. Most recently, Fang et al. [11] analyzed an unsteady flow over a stretching cylinder with expanding diameter. Due to the expansion of the cylinder, reversal flows were found in the results. In this work, we investigate the flow purely induced by unsteady expansion or contraction of a cylinder. As will be shown in the following, the current results are obtained based on similarity transformation and are exact solutions of the unsteady NS equations in a cylindrical coordinate system.

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2. Mathematical formulation

Consider the laminar viscous flow over a cylinder or a tube with a time dependent diameter (either contracting or expanding). Meanwhile, we assume there is no azimuthal velocity component. For incompressible fluids without body force and based on the axisymmetric flow assumption, the three-dimensional unsteady NS equations in cylindrical coordinates read [12]

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_z}{\partial z}=0, \tag{1}$$

$$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+u_z\frac{\partial u_r}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}+\frac{\partial^2 u_r}{\partial z^2}-\frac{u_r}{r^2}\right) \tag{2a}$$

$$\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+u_z\frac{\partial u_z}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{\partial^2 u_z}{\partial z^2}\right) \tag{2b}$$

where the velocity vector is $\vec{V}=(u_r,u_z)$, ν is the kinetic viscosity, p is the fluid pressure, and ρ is the fluid density. Due to the symmetry in the azimuthal direction, there are only two components in the cylindrical coordinates, say r and z. A schematic of the flow configuration is illustrated in Fig. 1. In this work, we assume the cylinder diameter is varying as a function of time $a(t)=a_0\sqrt{1-\beta t}$. For a positive value of β, the cylinder radius becomes smaller with time, e.g., contracting; while for a negative value of β, the diameter becomes larger with time, e.g., expanding. The boundary conditions (BCs) are

$$u_z(a(t),z,t)=0,\quad u_r(a(t),z,t)=-\frac{a_0\beta}{2\sqrt{1-\beta t}},\quad u_z(\infty,z,t)=0, \tag{3a—3c}$$

where β is a constant showing the expansion/contraction strength, $a(t)$ is the unsteady radius of the cylinder wall. The governing equations can be converted into a nonlinear ordinary differential equation with the following similarity transformation group,

$$u_z(r,z,t)=\frac{1}{a_0^2}\frac{4\nu z}{1-\beta t}f'(\eta),\quad u_r(r,z,t)=-\frac{1}{a_0}\frac{2\nu}{\sqrt{1-\beta t}}\frac{f(\eta)}{\sqrt{\eta}},\quad \text{and}\quad \eta=\left(\frac{r}{a_0}\right)^2\frac{1}{1-\beta t}. \tag{4a—4c}$$

Based on the defined velocity components, it is straightforward to derive from Eq. (2a) that the pressure gradient $\partial p/\partial r$ is a function of time t and r, and is independent on z. That is, $\partial p/\partial r = F(t,r)$ and $p=\int F(t,r)dr+G(t,z)$ where $G(t,z)$ is the constant of the integration. Therefore, it can be derived that $\partial p/\partial z = \partial G(t,z)/\partial z$. Hence, $\partial p/\partial z$ is independent on r. Then evaluating Eq. (2b) at $r\to\infty$ yields $\partial p/\partial z = 0$. Then by substituting the velocities components into Eq. (2b) and rearranging terms, Eq. (2b) can be transformed into a similarity equation as follows

$$\eta f'''+f''+ff''-f'^2-S(\eta f''+f')=0 \tag{5}$$

with BCs (3a–3c) transformed into the following

$$f(1)=S,\quad f'(1)=0,\quad f'(\infty)=0, \tag{6a—6c}$$

where $S=\frac{\beta a_0^2}{4\nu}$ is the unsteadiness parameter for the expanding/contracting cylinder showing the strength of expansion or contraction. Increasing the magnitude of S means a faster cylinder expansion or contraction. This parameter also indicates the ratio of a defined cylinder radius variation speed (βa_0) to the viscous diffusion speed (ν/a_0) like a Reynolds number because S can be rewritten as $S=\frac{\beta a_0}{4\nu/a_0}$. Based on previous discussion, we have shown that the fluid pressure does not depend on z. The pressure can be found from Eq. (2a) as

$$\frac{p}{\rho}=const+\nu\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)-\frac{1}{2}u_r^2+\int\frac{\partial u_r}{\partial t}dr. \tag{7}$$

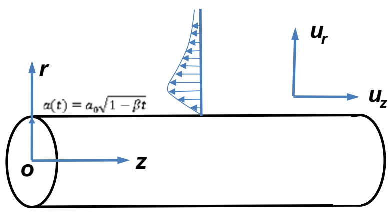

Fig. 1. Schematic of flow over an expanding cylinder with time dependent radius.