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Positive solutions of singular Caputo fractional differential equations with integral boundary conditions ☆

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ABSTRACT

In this paper, we investigate the existence of positive solutions of singular super-linear (or sub-linear) integral boundary value problems for fractional differential equation involving Caputo fractional derivative. Necessary and sufficient conditions for the existence of $C^3[0,1]$ positive solutions are given by means of the fixed point theorems on cones. Our nonlinearity $f(t,x)$ may be singular at $t = 0$ and/or $t = 1$.

1. Introduction

The singular ordinary differential equations arise in the fields of gas dynamics, Newtonian fluid mechanics, the theory of boundary layer and so on. The theory of singular boundary value problems has become an important area of investigation in recent years (see [1–6] and the references therein).

Differential equations of fractional order occur more frequently in different research areas and engineering, such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, control of dynamical systems, etc. Recently, many researchers paid attention to existence result of solution of the initial value problem and boundary value problem for fractional differential equations, such as [7–17]. Some recent contributions to the theory of fractional differential equations can be seen in [18–24].

In [24], by using a fixed-point theorem on cones, Zhang investigated the existence and multiplicity results of positive solutions for boundary value problem of fractional order differential equation

$$\begin{aligned}&{}^{C}D_{0+}^{p}u(t) = f(t,u(t)), \quad 0 < t < 1,\\ &u(0) + u'(0) = 0, \quad u(1) + u'(1) = 0,\end{aligned}$$

where $1 < p \leqslant 2$ is a real number, and ${}^{C}D_{0+}^{p}$ is the Caputo fractional derivative, and f: $[0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous.

Yang in [25,26] deals with the existence and nonexistence of positive solutions for the second order differential equation with integral boundary value problem by means of a fixed point theorem in a cone. Wei and Zhang in [27] investigate the existence of positive solutions for fourth order singular p-Laplacian differential equations with integral boundary conditions,

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and give a necessary and sufficient condition for the existence of $C^2[0,1]$ as well as $pseudo - C^3[0,1]$ positive solutions by means of constructing lower and upper solutions.

The purpose of this paper is to consider the existence of the positive solution of singular super-linear (or sub-linear) integral boundary value problems for differential equations of fractional order,

$$\begin{cases} {}^{C}D^{\alpha}_{0+}x(t) + f(t, x(t)) = 0, \quad t \in (0,1), \\ ax(0) - bx'(0) = 0, \quad cx(1) + dx'(1) = 0, \\ x''(0) + x'''(0) = \int_0^1 x''(\tau)dp(\tau), \quad x''(1) + x'''(1) + \int_0^1 x''(\tau)dq(\tau) = 0, \end{cases} \tag{1.1}$$

where $3 < \alpha \leqslant 4$ is a real number, $a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$, $d \geqslant 0$, $\rho = ad + ac + bc > 0$. ${}^{C}D^{\alpha}_{0+}$ is the Caputo fractional derivative and f satisfies the following either

(**H1**): $f \in C((0,1) \times [0,\infty), [0,\infty))$, $f(t,(at+b)(c(1-t)+d)) > 0$, $t \in (0,1)$, and there exist constants λ, μ $(1 < \lambda \leqslant \mu < \infty)$, such that for $t \in (0,1)$, $x \in (0,\infty)$

$$r^{\mu} f(t,x) \leqslant f(t,rx) \leqslant r^{\lambda} f(t,x), \quad \text{if } 0 < r \leqslant 1; \tag{1.2}$$

or

(**H2**): (i) $f(t,x)$ is nonnegative continuous on and increasing on x, it may be singular at $t = 0$ and/or $t = 1$; (ii) there exists $\sigma \in (0,1)$ such that

$$f(t,rx) \geqslant r^{\sigma} f(t,x), \quad \forall 0 < r \leqslant 1, \quad (t,x) \in (0,1) \times [0,+\infty). \tag{1.3}$$

(**H3**): $p(t)$ and $q(t)$ are right continuous on $[0,1)$, left continuous at $t = 0$, and nondecreasing on $[0,1]$, with $p(0) = q(0) = 0$; $\int_0^1 x(\tau)dp(\tau)$ and $\int_0^1 x(\tau)dq(\tau)$ denote the Riemann–Stieltjes integrals of x with respect to p and q, respectively, and

$$0 \leqslant \int_0^1 dp(\tau) < 1 \quad \text{and} \quad 0 \leqslant \int_0^1 dq(\tau) < 1, \tag{1.4}$$

$$\frac{2 + \int_0^1 \tau dq(\tau)}{1 + \int_0^1 dq(\tau)} > \frac{1 - \int_0^1 \tau dp(\tau)}{1 - \int_0^1 dp(\tau)}. \tag{1.5}$$

Remark 1.1. (1.2) implies

$$r^{\lambda} f(t,x) \leqslant f(t,rx) \leqslant r^{\mu} f(t,x), \quad \text{if } r \geqslant 1. \tag{1.6}$$

Conversely, (1.6) implies (1.2).

Remark 1.2. (1.3) implies

$$f(t,rx) \leqslant r^{\sigma} f(t,x), \quad \forall r \geqslant 1, \quad (t,x) \in (0,1) \times [0,+\infty). \tag{1.7}$$

Conversely, (1.7) implies (1.3).

Typical functions that satisfy the above super-linear (1.2) (or sub-linear (1.3)) hypothesis are those taking the form $f(t,x) = \sum_{k=1}^{n} p_k(t)x^{\lambda_k}$ $\left(f(t,x) = \sum_{k=1}^{n} p_k(t)x^{q_k}\right)$; here $p_k(t) \in C(0,1)$, $p_k(t) > 0$ on $(0,1)$, $\lambda_k > 1$ (or $0 < q_k < 1$), $k = 1,2,\ldots,n$.

By singularity we mean that the function f in (1.1) is allowed to be unbounded at the end points $t = 0$ and $t = 1$. A function $x(t) \in C^2[0,1] \cap C^4(0,1)$ is called a $C^2[0,1]$ (positive) solution of (1.1) if it satisfies (1.1) ($x(t) > 0$, $x''(t) < 0$ for $t \in (0,1)$). $AC^2[0,1]$ (positive) solution of (1.1) is called a $C^3[0,1]$ (positive) solution if $x^{(3)}(0^+)$ and $x^{(3)}(1^-)$ both exist ($x(t) > 0$, $x''(t) < 0$ for $t \in (0,1)$).

For the case of integer order: $\alpha = 4$, a sufficient condition for the existence of solutions of the singular problem (1.1) was given by O'Regan in [4] with a topological transversal Theorem; In the case of sub-linear, a sufficient and necessary condition for the existence of $C^2[0,1]$ as well as $C^3[0,1]$ positive solutions of the singular problem (1.1) was given by Wei in [6] with the method of lower and upper solutions and with the comparison theorem.

While for the case of fractional order: $3 < \alpha \leqslant 4$, the sufficient and necessary condition for the existence of positive solutions of the singular problem (1.1) has not been given up to now, the research proceeds slowly and appears some new difficulties in obtaining the sufficient and necessary conditions.

Now, in this paper, we shall give necessary and sufficient conditions for the existence of $C^3[0,1]$ positive solutions of the singular super-linear (or sub-linear) problem (1.1) by using the fixed point theorems on cones. Our nonlinearity $f(t,x)$ may be singular at $t = 0$ and/or $t = 1$.

2. Preliminaries and lemmas

In this section, we present some definitions and establish some lemmas.

Let $J = [0,1]$ be a compact interval on the real axis $\mathbb{R}$, and y be a measurable Lebesgue function, that is, $y \in L_1(0,1)$. Let $t \in J$ and $\alpha \in \mathbb{R}$ $(\alpha > 0)$, and