Commun Nonlinear Sci Numer Simulat 17 (2012) 3328–3344

Contents lists available at SciVerse ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Symbolic computation of normal form for Hopf bifurcation in a retarded functional differential equation with unknown parameters

Li Zhang [a], Huailei Wang [a], Haiyan Hu [a,b,*]

[a] State Key Laboratory of Mechanical Vibration and Strength, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, People's Republic of China
[b] School of Aerospace Engineering, Beijing Institute of Technology, Beijing 100081, People's Republic of China

ARTICLE INFO

Article history:
Received 11 July 2011
Received in revised form 28 November 2011
Accepted 30 November 2011
Available online 13 December 2011

Keywords:
Delay differential equation
Normal form
Center manifold
Symbolic computation
Maple program

ABSTRACT

Based on the normal form theory for retarded functional differential equations by Faria and Magalhães, a symbolic computation scheme together with the Maple program implementation is developed to compute the normal form of a Hopf bifurcation for retarded functional differential equations with unknown parameters. Not operating as the usual way of computing the center manifold first and normal form later, the scheme features computing them simultaneously. Great efforts are made to package this task into one Maple program with an input interface provided for defining different systems. The applicability of the Maple program is demonstrated via three kinds of delayed dynamic systems such as a delayed Liénard equation, a simplified drilling model and a delayed three-neuron model. The effectiveness of Maple program is also validated through the numerical simulations of those three systems.

1. Introduction

Memory effects, hereditary effects and time lag effects are intrinsically similar features of many dynamical systems. They give rise to an irreducible influence of the past history on the evolution of a dynamic system. In this context, a deviated time-argument is usually introduced and embodied as the term of time delay in mathematical modeling of such a delayed dynamic system. Generally speaking, differential equations with time delays belong to the category of Functional Differential Equations (FDEs) [1,2], where, for clarity, the past dependence through only state variables leads to Retarded Functional Differential Equations (RFDEs) or Retarded Delay Differential Equations (RDDEs), while the past dependence on the derivative of the state variables yields Neutral Functional Differential Equations (NFDEs) or Neutral Delay Differential Equations (NDDEs) [3]. As opposed to Ordinary Differential Equations (ODEs), the state space of FDEs is infinite-dimensional. This fact implies an exceeding complexity in both analytical and numerical treatments.

To study the local dynamic behavior of a system around its equilibrium point, the center manifold reduction and the normal form transformation have been proved to be very effective. At a bifurcation point, the flow near the equilibrium is essentially governed by the vector field on the finite dimensional center manifold. With the center manifold reduction, the original FDEs can be reduced to ODEs. Moreover, after the normal form implementation, where successive, near-identity, nonlinear transformations are used, a simpler form can be obtained for the purpose of simplifying the dynamical analysis [4]. These two methods have been widely applied to studying the local bifurcations of delayed dynamic systems. For example, Xu and Lu studied the Hopf bifurcation of delayed Liénard equations [5]. Stone and Campbell presented an analysis on the Hopf

* Corresponding author at: State Key Laboratory of Mechanical Vibration and Strength, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, People's Republic of China. Tel.: +86 25 84891672; fax: +86 25 84891512.

E-mail address: hhyae@nuaa.edu.cn (H. Hu).

1007-5704/$ - see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.cnsns.2011.11.035

bifurcation for a nonlinear Delay Differential Equation (DDE) modeling a drilling process [6]. Liao et al. considered the Hopf bifurcation analysis in a Tri-neuron model with a time delay [7]. In the above mentioned researches, a center manifold was computed in advance and then the calculation of normal form was followed. Faria and Magalhães provided an alternative method for obtaining the normal form of an RFDE by considering the RFDE as an ODE in an appropriate infinite-dimensional phase space so that the center manifold needs not to be computed beforehand [8,9]. In summary, there are mainly two possible approaches to the normal form analysis for a delayed dynamic system as follows:

(A) Compute the center manifold and project the flow of the system on the center manifold first, and then compute the corresponding normal form [5–7].
(B) Conduct the center manifold reduction and the normal form analysis simultaneously [8,9].

Although the normal form is a very effective tool for studying the local bifurcations of FDEs, the corresponding procedures are usually very complicated and tedious. Fortunately, with the development of computer algebra systems, such as Maple and Mathematica, many efforts have been made to develop symbolic procedures so that the normal form can be easily computed [10,11]. For example, Belair and Campell presented a symbolic algebra algorithm to approximate the center manifold for DDEs [12,13]. Babram et al. addressed the algorithms for computing the terms of the center manifold for RFDEs and NFDEs [14,15]. Qesmi et al. developed symbolic procedures for computing the center manifold for the Hopf bifurcation, the Bogdanov–Takens bifurcation and the Fold–Hopf bifurcation for RFDEs [16–18]. All the above mentioned algorithms or procedures are based on the approach (A), which implies that the center manifold needs to be computed first. Babram mentioned a computation scheme for RFDEs without bifurcation parameters according to the approach (B) [19]. In [16–18], Qesmi et al. argued that it is difficult to apply approach (B) to computing the normal forms of higher orders because the approach has the necessity of solving a functional differential equation. However, the functional differential equation can be transformed as an initial value problem. With an increasing power of computer algebra systems, the approach (B) could be automatically implemented.

This study aims at developing a Maple program based on the approach (B) so as to compute the normal form of a Hopf bifurcation for an RFDE with unknown parameters as follows

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{L}(\alpha)\boldsymbol{x}_t + \boldsymbol{F}(\boldsymbol{x}_t, \alpha), \tag{1}$$

with $\boldsymbol{x}_t \in \boldsymbol{C}_n = \boldsymbol{C}([-r, 0]; \mathbb{R}^n), \boldsymbol{x}_t(\theta) = \boldsymbol{x}(t+\theta), \alpha \in \mathbb{R}^p (p = 1), \alpha \mapsto \boldsymbol{L}(\alpha)$ is a C^∞ function with values in the space of bounded linear operators which project the Banach space $\boldsymbol{C}_n = \boldsymbol{C}([-r, 0]; \mathbb{R}^n)$ to $\mathbb{R}^n$, $\boldsymbol{L}(0)\boldsymbol{x}_t := \boldsymbol{L}_0\boldsymbol{x}_t = \int_{-r}^{0} d[\boldsymbol{\eta}(\theta)]\boldsymbol{x}_t(\theta)$, where $\boldsymbol{\eta}$ is an $n \times n$ matrix-valued function of bounded variation defined on $[-r,0]$. and $\boldsymbol{F}$ is a C^∞ function from $\boldsymbol{C}_n \times \mathbb{R}^p$ to $\mathbb{R}^n$ with $\boldsymbol{F}(\boldsymbol{0}, \alpha) = \boldsymbol{0}$ and $\boldsymbol{F}'(\boldsymbol{0}, \alpha) = \boldsymbol{0}$ for all $\alpha \in \mathbb{R}^p$, where $\boldsymbol{F}'$ denotes the Frechet derivatives of $\boldsymbol{F}$. The bifurcation parameter α is taken into account during the normal form transformation so that no further unfolding for the parameter is needed [20].

As the Hopf bifurcation is considered in the study, the following hypothesis is assumed:

(**H**) The characteristic equation of Eq. (1) has a simple pair of eigenvalues $\gamma(\alpha) \pm i\omega(\alpha)$, which cross the imaginary axis transversely at $\alpha = 0$ so that the following conditions hold

$$\gamma(0) = 0, \quad \omega(0) = \omega_c > 0, \quad \left.\frac{d\gamma(\alpha)}{d\alpha}\right|_{\alpha=0} \neq 0$$

and no other eigenvalue has zero real part.

The paper is organized as follows. In Section 2, some elementary notions and the theoretical background of RFDEs (for more detail see [8]) are provided. The normal form transformation for an RFDE is implemented in Section 3 and the corresponding computation scheme is outlined in Section 4. In Section 5, the powerfulness of the Maple program is demonstrated via three kinds of delayed dynamical systems, including a delayed Liénard system, a simplified drilling system and a delayed three-neuron system. The correctness of the Maple program is verified through the comparisons with the numerical simulations of those three delayed dynamic systems. Some conclusions are drawn in Section 6.

2. Theoretical background

The linearized part of Eq. (1) takes the form of

$$\frac{d}{dt}\boldsymbol{x}_t = \boldsymbol{L}_0\boldsymbol{x}_t \tag{2}$$

and its solution defines a C^0 semigroup $\boldsymbol{T}(t)$ on $\boldsymbol{C}_n$, $\boldsymbol{T}(t)\boldsymbol{\phi} = \boldsymbol{x}_t(\boldsymbol{\phi})$, $t \geqslant 0$. The infinitesimal generator $\boldsymbol{A}_0$, which is associated with $\boldsymbol{T}(t)$, is given by $\boldsymbol{A}_0\boldsymbol{\phi} = \dot{\boldsymbol{\phi}}$. $\boldsymbol{A}_0$ has the domain

$$D(\boldsymbol{A}_0) = \left\{ \boldsymbol{\phi} \in \boldsymbol{C}_n^1 : \left.\frac{d\boldsymbol{\phi}}{d\theta}\right|_{\theta=0} = \boldsymbol{L}_0\boldsymbol{\phi} \right\},$$