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# An optimal homotopy-analysis approach for strongly nonlinear differential equations

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ABSTRACT

In this paper, an optimal homotopy-analysis approach is described by means of the nonlinear Blasius equation as an example. This optimal approach contains at most three convergence-control parameters and is computationally rather efficient. A new kind of averaged residual error is defined, which can be used to find the optimal convergence-control parameters much more efficiently. It is found that all optimal homotopy-analysis approaches greatly accelerate the convergence of series solution. And the optimal approaches with one or two unknown convergence-control parameters are strongly suggested. This optimal approach has general meanings and can be used to get fast convergent series solutions of different types of equations with strong nonlinearity.

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## 1. Introduction

Nonlinear equations are much more difficult to solve than linear ones, especially by means of analytic methods. Generally speaking, there are two standards for a *satisfactory* analytic method of nonlinear equations:

(a) it can *always* give approximation expressions *efficiently*;
(b) it can guarantee that approximation expressions are *accurate* enough in the *whole* region of *all* physical parameters.

Using above two standards as a criterion, we can discuss the advantages and disadvantages of different analytic techniques for nonlinear problems.

Perturbation techniques [1–6] are widely applied in science and engineering. Most perturbation techniques are based on small (or large) physical parameters in governing equations or boundary conditions, called perturbation quantities. In general, perturbation approximations are expressed in a series of perturbation quantities, and the original nonlinear equations are replaced by an infinite number of linear (sometimes even nonlinear) sub-problems, which are completely determined by the original governing equation and especially by the place where perturbation quantities appear. Perturbation methods are simple, and easy to understand. Especially, based on small physical parameters, perturbation approximations often have clear physical meanings. Unfortunately, *not* every nonlinear problem has such kind of perturbation quantity. Besides, even if there exists a small parameter, the sub-problem might have no solutions, or might be rather complicated so that only a few of the sub-problems can be solved. Thus, it is *not* guaranteed that one can always get perturbation approximations efficiently for any a given nonlinear problem. More importantly, it is well-known that most perturbation approximations are valid only

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for small physical parameters. In general, it is not guaranteed that a perturbation result is valid in the whole region of all physical parameters. Thus, perturbation techniques do not satisfy not only the standard (a) but also the standard (b) mentioned at the beginning of this section.

To overcome the restrictions of perturbation techniques, some traditional nonperturbation methods are developed, such as Lyapunov's artificial small parameter method [7], the $\delta$-expansion method [8,9], Adomian decomposition method [10–15], and so on. In principle, all of these methods are based on a so-called artificial parameter, and approximation solutions are expanded into series of such kind of artificial parameter. This artificial parameter is often used in such a way that one can get approximation solutions efficiently for a given nonlinear equation. Compared with perturbation techniques, this is indeed a great progress. However, in theory, one can put the artificial small parameter in many different ways, but unfortunately there are no theories to guide us how to put it in a better place so as to get a better approximation. For example, Adomian decomposition method simply uses the linear operator $d^k/dx$ in most cases, where $k$ is the highest order of derivative of governing equations, and therefore it is rather easy to get solutions of the corresponding sub-problems by means of integration $k$ times with respect to $x$. However, such simple linear operator gives approximation solutions in power-series, but unfortunately power-series has often a finite radius of convergence. Thus, Adomian decomposition method cannot ensure the convergence of its approximation series. Generally speaking, all traditional nonperturbation methods, such as Lyapunov's artificial small parameter method [7], the $\delta$-expansion method [8,9] and Adomian decomposition method [10–15], can *not* guarantee the convergence of approximation series. So, these traditional nonperturbation methods satisfy only the standard (a) but *not* the standard (b) mentioned before.

In 1992 Liao [16] took the lead to apply the homotopy [17], a basic concept in topology [18], to get analytic approximations of nonlinear differential equations. Liao [16] described the early form of the homotopy-analysis method (HAM) in 1992. For a given nonlinear differential equation

$$\mathcal{N}[u(x)] = 0, \quad x \in \Omega,$$

where $\mathcal{N}$ is a nonlinear operator and $u(x)$ is a unknown function, Liao constructed a *one-parameter* family of equations in the embedding parameter $q \in [0, 1]$, called the zeroth-order deformation equation

$$(1-q)\mathcal{L}[U(x;q) - u_0(x)] + q\mathcal{N}[U(x;q)] = 0, \quad x \in \Omega, \;\; q \in [0,1], \tag{1}$$

where $\mathcal{L}$ is an auxiliary linear operator and $u_0(x)$ is an initial guess. The homotopy provides us larger freedom to choose both of the auxiliary linear operator $\mathcal{L}$ and the initial guess than the traditional nonperturbation methods mentioned before, as pointed out later by Liao [19–21]. At $q = 0$ and $q = 1$, we have $U(x;0) = u_0(x)$ and $U(x;1) = u(x)$, respectively. So, if the Taylor series

$$U(x;q) = u_0(x) + \sum_{n=1}^{+\infty} u_n(x) q^n \tag{2}$$

converges at $q = 1$, we have the so-called homotopy-series solution

$$u(x) = u_0(x) + \sum_{n=1}^{+\infty} u_n(x), \tag{3}$$

which must satisfy the original equation $\mathcal{N}[u(x)] = 0$, as proved by Liao [19,20] in general. Here, $u_n(x)$ is governed by a linear differential equation related to the auxiliary linear operator $\mathcal{L}$ and therefore is easy to solve, as long as we choose the auxiliary linear operator properly. In some cases, one can get convergent series of nonlinear differential equations by choosing proper linear operator and initial guess. However, Liao [22,20] found that this early homotopy-analysis method can not always guarantee the convergence of approximation series. To overcome this restriction, Liao [22] in 1997 introduced such a nonzero auxiliary parameter $c_0$ to construct a *two-parameter* family of equations, i.e. the zeroth-order deformation equation.[1]

$$(1-q)\mathcal{L}[U(x;q) - u_0(x)] = c_0 q \mathcal{N}[U(x;q)], \quad x \in \Omega, \;\; q \in [0,1]. \tag{4}$$

In this way, the homotopy-series solution (3) is not only dependent upon $x$ but also the auxiliary parameter $c_0$. It was found [22,19,20] that the auxiliary parameter $c_0$ can adjust and control the convergence region and rate of homotopy-series solutions. In essence, the use of the auxiliary parameter $c_0$ introduces us one more "artificial" degree of freedom, which has no physical meaning but greatly improved the early homotopy-analysis method: it is the auxiliary parameter $c_0$ which provides us a convenient way to guarantee the convergence of homotopy-series solution [22,20]. Currently, Liang and Jeffrey [23] used a simple example to illustrate the importance of the auxiliary parameter $c_0$. Besides, Liao [24] revealed the relationship between the homotopy-analysis method (in some special cases) and the famous Euler transform, which explains clearly why the homotopy-analysis method can ensure the convergence of homotopy-series solution. Due to this reason, $c_0$ was renamed currently as the *convergence-control parameter* [25].

[1] Liao [22] originally used the symbol $\hbar$ to denote the auxiliary parameter. But, $\hbar$ is well-known as Planck's constant in quantum mechanics. To avoid misunderstanding, we suggest to use the symbol $c_0$ to denote the "basic" convergence-control parameter.

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