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Relationship between the homotopy analysis method and harmonic balance method

Y.M. Chen [a,*], J.K. Liu [b], G. Meng [a]

[a] State Key Laboratory of Mechanical System and Vibration, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai 200240, China
[b] Department of Mechanics, Sun Yat-sen University, Guangzhou 510275, China

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A B S T R A C T

This paper presents a study of the relationship between the homotopy analysis method (HAM) and harmonic balance (HB) method. The HAM is employed to obtain periodic solutions of conservative oscillators and limit cycles of self-excited systems, respectively. Different from the usual procedures in the existing literature, the HAM is modified by retaining a given number of harmonics in higher-order approximations. It is proved that as long as the solution given by the modified HAM is convergent, it converges to one HB solution. The Duffing equation, the van der Pol equation and the flutter equation of a two-dimensional airfoil are taken as illustrations to validate the attained results.

1. Introduction

Over the past decade, Liao described a nonlinear analytical technique which does not require small parameters and thus can be applied to solve nonlinear problems without small or large parameters [1–4]. This technique is based on homotopy theory, which is an important part of topology, thus called the homotopy analysis method (HAM). Its fundamental idea is to construct a class of homotopy in a rather general form by introducing an auxiliary parameter, through which nonlinear problems can be transformed into a series of linear sub-problems. The auxiliary parameter can provide us with a convenient way to control the convergence of approximation series and adjust convergence regions when necessary. The systematical description of this method was given in Ref. [5]. Also in this paper, the author discussed the convergence of the solution series and showed that as long as the series given by HAM converges, it must converge to one solution of the nonlinear problem under consideration. In the rapid development of HAM, it has been widely used in various nonlinear problems [6–10].

Perturbation method [11] is one of the most widely applied analytic tools for nonlinear problems. Essentially, perturbation techniques are based on the existence of a small/large parameter or variable, which is often called perturbation quantity. The existence of perturbation quantities, however, is a cornerstone of these techniques. The dependence of perturbation techniques on small/large parameters might be avoided by introducing a so-called artificial small parameter, such as the Lyapunov artificial small parameter method [12], the δ-expansion method [13] and the Adomian's decomposition method [14]. Liao [15] proved that they are all special cases of the HAM, which implies the HAM is more generalized. Additionally,

* Corresponding author.
E-mail address: chen.yan.mao@hotmail.com (Y.M. Chen).

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exploring the inner relationship between existing computational techniques is of fundamental interest to many researchers engaged in computing science. Thus, it is worth and interesting to investigate its relationship of the HAM to other methods for nonlinear systems.

The main aim of this paper is to study the relationship between the HAM and harmonic balance (HB) method. The basic procedure of the HB method is to transform the problem under consideration into a set of nonlinear algebraic equations by describing the possible periodic/limit cycle solution as truncated Fourier series [16]. That means the solutions given by the HB method possess a limited number of harmonics. However, the highest harmonic of the periodic solutions obtained by the HAM increases unboundedly [6]. For this issue, the HAM is slightly modified by retaining several lower-order harmonics to obtain solutions in the same form of HB ones. A major finding of this paper is that as long as the solution given by the modified HAM converges, it must converge to one HB solution. In order to validate it, proofs are given and three numerical examples are also presented.

2. Homotopy analysis method

Consider a nonlinear autonomous system described by

$$f(x,\dot{x},\ddot{x}) = 0 \tag{1}$$

where the superscript denotes the differentiation with respect to time t. In this study, system (1) may either be a conservative or self-excited system so that it possesses at least one periodic (or limit cycle) solution. Introducing a new time scale

$$\tau = \omega t \tag{2}$$

where ω is the angular frequency of the possible periodic solution, then (1) becomes

$$f(x,\omega x',\omega^2 x'') = 0 \tag{3}$$

where the superscript denotes the differentiation with respect to τ.

2.1. Self-excited system

In general, limit cycles of self-excited oscillating systems contain two important physical parameters, i.e., the frequency ω and the amplitude a of oscillation. They are both independent upon the initial conditions. Without loss of generality, consider simple initial conditions

$$x(0) = a, \quad x'(0) = 0. \tag{4}$$

For self-excited system, a is the amplitude of the limit cycle to be determined.

According to Eqs. (3) and (4), let

$$x_0(\tau) = a_0 \cos(\tau) \tag{5}$$

be the initial guess of $x(\tau)$, where a_0 is the one of a. Likewise, let ω_0 be the initial approximation of ω. The HAM is based on such continuous variations $\phi(\tau,p)$, $\Omega(p)$ and $A(p)$ that, as the embedding parameter p increases from 0 to 1, $\phi(\tau,p)$ varies from the initial guess $x_0(\tau) = a\cos\tau$ to the exact solution $x(\tau)$, so do $\Omega(p)$ and $A(p)$ from ω_0 and a_0 to ω and a, respectively.

The rule of solution expression [6] states that $x(\tau)$ can be described as a set of base functions $\{\cos(k\tau), \sin(k\tau) | \ k = 0,1,2,\ldots\}$, based on which we can choose such an auxiliary linear operator

$$L[\phi(\tau,p)] = \frac{\partial^2\phi(\tau,p)}{\partial\tau^2} + \phi(\tau,p) \tag{6}$$

so that

$$L[\cos\tau] = L[\sin\tau] = 0. \tag{7}$$

Then according to Eq. (3), we define the following nonlinear operator:

$$\Psi[\phi(\tau,p),\Omega(p),A(p)] = f\left[\phi(\tau,p),\Omega(p)\frac{\partial\phi(\tau,p)}{\partial\tau},\Omega^2(p)\frac{\partial^2\phi(\tau,p)}{\partial^2\tau}\right] \tag{8}$$

where $p \in [0,1]$ is the embedding parameter. Letting h be a nonzero auxiliary parameter, we construct such a homotopy in a general form

$$H[\phi(\tau,p);h,p] = (1-p)L[\phi(\tau,p) - x_0(\tau)] - hp\Psi[\phi(\tau,p),\Omega(p),A(p)]. \tag{9}$$

Setting $H[\phi(\tau,p);h,p] = 0$ yields a family of equations

$$(1-p)L[\phi(\tau,p) - x_0(\tau)] = hp\Psi[\phi(\tau,p),\Omega(p),A(p)] \tag{10}$$

subject to the initial conditions