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Estimate the shortest paths on fractal m-gons

Mohamed Mubarak *

Mathematics Department, Faculty of Science, Minia University, El-Minia 61519, Egypt
Mathematics Department, University College in Lieth, Umm Al-Qura University, El-Lieth 21961 P. Box 112, Saudi Arabia

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The self-similar sets seem to be a class of fractals which is most suitable for mathematical treatment. The study of their structural properties is important. In this paper, we estimate the formula for the mean geodesic distance of self-similar set (denote fractal m-gons). The quantity is computed precisely through the recurrence relations derived from the self-similar structure of the fractal considered. Out of result, obtained exact solution exhibits that the mean geodesic distance approximately increases as a exponential function of the number of nodes (small copies with the same size) with exponent equal to the reciprocal of the fractal dimension.

1. Introduction

Most of the classical fractals are self-similar sets [1–3] with certain symmetries. One of these sets is the well-known Sierpinski gasket. Hinz and Schief [4], calculated the average interior distance for this fractal. In a subset F of R^n the interior distance between two points $x, y \in F$ is usually defined as the shortest possible length of a curve inside F connecting x and y. This definition also works for some fractals similar to Sierpinski gasket. The average interior distance can be seen as a problem of geometrical probability. For certain sets with fractional dimension γ, we will replace Lebesque measure by the γ-dimensional Housdorff measure. Hinz and Schief used measure theory to determine the average interior distance. Bandt and Mubarak [5], determined the distribution of Euclidean and interior distances in the Sierpinski gasket. The problem of the shortest paths with F is a bit more intricate. Usually, these shortest paths will be fractals themselves [6,7,1]. An analytical theory including Brownian motion [8], Dirichlet forms [9], spectrum of the Laplacian [10] and geodesics [7] have been developed only on symmetric fractals.

2. Structure of the fractal *m*-gons

A compact subset F of the complex plan is called a fractal m-gons if F fulfils equation

$$F = \psi_1(F) \cup \cdots \cup \psi_m(F) \tag{1}$$

for similarity mappings $\psi_k(z) = \lambda_k z + c_k$: $k = 1, 2, \ldots, m$, and there is a rotation in the plane which acts transitively on the pieces, permuting them in an m-cycle. Fig. 1, shows examples which seem to be new [11]. For each m, all fractal m-gons are parameterized by one complex λ running through the unit disk [11,12]. Each point z with address $j_1, j_2, \ldots, j_k \in \{0, 1, \ldots, m-1\}$ in the fractal m-gons $F(\lambda)$ has the representation

* Address: Mathematics Department, Faculty of Science, Minia University, El-Minia 61519, Egypt. Tel.: +20 862335716.
E-mail addresses: mubarak_7061111@yahoo.com, mamubarak@uqu.edu.sa

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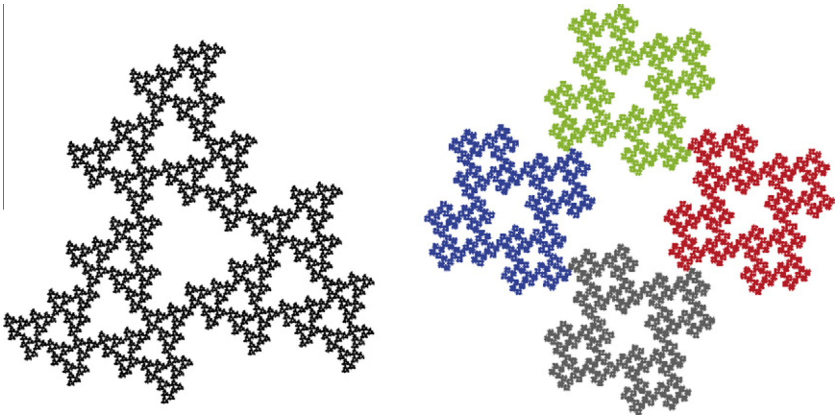

Fig. 1. The left self-similar set with $\lambda_F = 1.8011 + i.\,0.6656 = (|\lambda|,\theta) \approx (1.920015, +20.28°)$. The right self-similar set with $\lambda_F = 2 - i = (|\lambda|,\theta) \approx (2.2361, -26.57)$. The neighboring pieces intersect in one point, $F_{k+1} \cap F_k = \{P_{k,k+1}\}: k+1 \mod m,\quad m = 3,4$.

$$z = \sum_{k=1}^{\infty} b^{j_k}\ \lambda^{k-1} \tag{2}$$

Thus $F(\lambda)$ is the support of a random series of powers of λ with coefficients chosen from the m-roots of unity. Therefore,

$$\psi_{j_1} \circ \cdots \circ \psi_{j_k}(0) = b^{j_1} + \lambda b^{j_2} \cdots + \lambda^{k-1} b^{j_k}. \tag{3}$$

In [11–14] there are various explanations of how to determine λ for the fractal in Fig. 1. A self-similar structure with m pieces F_ω is connected only if the graph with vertices $\omega = 1,\ldots,m$ and with edges $\{i,j\}$ for intersecting pieces $F_i \cap F_j \neq \emptyset$ is connected [6,15]. We shall see that, this also holds for fractal m-gons, see proposition 1 in [12]. Besides, we say that an m-gons F is simple if $F_0 \cap F_1$ is a singleton with two addresses of the form $0\bar{i} \sim 1\bar{j}$ (see Sierpinski gasket). A central issue in the study of complex system is to understand how their dynamical behaviors are influenced by underlying geometrical and topological properties [16,17]. Among's many fundamental structural characteristics [18,19], mean geodesic distance is an important topological feature of complex system that are often described by graphs (or network) where nodes (vertices) represent the component units of system and links (edges) stand for the interaction between them [20,21].

The concept of the mean geodesic distance. Mean geodesic distance is defined as a mean length of the shortest paths between a pairs of nodes. The mean geodesic distance is directly related to many aspects of real systems, such as signal integrity in communication networks, the propagation of beliefs in social networks or of technology in industrial networks. Moreover, the shortest path has been found crucial for an efficient locally excited neuronal dynamics [22]. Similarly is the case in greater survivability of cooperation in social dilemma situations [23]. Recent studies indicated that a number of other dynamical processes are also relevant to mean geodesic distance, including disease spreading [24], random walk [25]. Undoubtedly, the geodesic distance is very important consequently in this present paper, we investigate this interesting quantity analytically with respect to fractal m-gons.

3. Construction of the main idea

Firstly, we consider the small pieces with the same size at t-level and its diameters δ_0 as nodes and edges respectively. We drive an exact formula for the mean geodesic characterizing the fractal m-gons. The analytic method is on the basis of an algebraic iterative procedure obtained from the self-similar structure of fractal m-gons. The obtained result shows that the mean geodesic distance exponentially with the number of nodes.

A well-known theorem of Hutchinson [26–28] says that for any finite set of contractions $\psi_i: i = 1, 2, \ldots, m$, there is exactly one invariant set F. If the ψ_i are similarities with factor r_i, which satisfy Eq. (1), then F is called a *self-similar set*, and the unique number $\gamma > 0$ with $\sum_i r_i^\gamma = 1$, is called the *similarity dimension* of F. Let F be an infinite locally finite connected self-similar structure. That is to say, F is a set whose elements are called small pieces at t-level (assuming nodes with respect network). If F is connected, that means that for every pair x, y of nodes F there is at least one path in F joining x and y. The contraction maps are then iterated until the length of each edge (diameter of small pieces) is approximately $\delta_0 = \lambda^{-t}$. A sequence of graphs G_t convergent to the fractal with the property that each edge has roughly the same length, and hence a random walker will move along any edge with roughly equal probability. It is worth mentioning that, the distance $d_t(x,y)$ is the length of the shortest path from x to y which denote $x \sim y$. A path from x to y is geodesic if its length is $d_t(x,y)$.