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Solving systems of ODEs by homotopy analysis method

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Abstract

This paper applies the homotopy analysis method (HAM) to systems of ordinary differential equations (ODEs). The systems investigated include stiff systems, the chaotic Genesio system and the matrix Riccati-type differential equation. The HAM gives approximate analytical solutions which are of comparable accuracy to the seven- and eight-order Runge–Kutta method (RK78).

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1. Introduction

The purpose of this paper is to find approximate solution to a system of ordinary differential equations (ODEs) of the form:

$$y_i' = f_i(t, y_1, \ldots, y_n), \quad y_i(t_0) = y_{0,i}, \quad i = 1, 2, \ldots, n, \tag{1}$$

where f_i are (linear or nonlinear) real-valued functions, $t_0 \in \Re$ and $y_{0,i} \in \Re$. By a change of independent variable $t \to t + t_0$, systems of the form (1) can always be translated to the origin, and so in this paper, we focus on finding approximate solution to equations of the form

$$y_i' = f_i(t, y_1, \ldots, y_n), \quad y_i(0) = y_{0,i}, \quad i = 1, 2, \ldots, n. \tag{2}$$

Many practical situations can be modelled by different types of system (2). We shall consider three different types of system (2). Many phenomena in chemical kinetics and engineering are modelled by stiff systems. (For the definition and more information on stiffness we refer the reader to [1].) Stiff problems pose special computational difficulties because explicit numerical methods cannot solve these problems without severe limitations on the step size. In control theory, systems of the form (2) can also exhibit chaotic behaviours [2,3]. As

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is well known, a chaotic system is a nonlinear deterministic system having complex and unpredictable behavior and exhibits sensitive dependence on initial conditions. One such system is the so-called Genesio system [4]. It is one of the paradigms of chaos since it captures many features of chaotic systems. The third important system we shall study is the Riccati-type systems [5] which are used widely in the engineering sciences and financial mathematics [6].

Various numerical integration algorithms (for example, Runge–Kutta algorithms) for approximating solutions of the above three types of systems (2) have been presented in the literature. However, these algorithms offer approximate solutions at discrete points only thereby making it impossible to get continuous solutions. Recently there has been a growing interest in obtaining continuous solutions to systems of the form (2) by analytical techniques. One such technique yielding series solutions is called the homotopy analysis method (HAM), initially proposed by Liao in his PhD thesis [7]. (For a systematic and clear exposition on HAM the reader is referred to [8].) In recent years, this method has been successfully employed to solve many types of problems in science and engineering [9–21]. Homotopy analysis method contains an auxiliary parameter h which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution.

This paper investigates for the first time the applicability and effectiveness of HAM on linear and nonlinear stiff systems and a chaotic system, and extends Abbasbandy's [19] HAM solution of a 1-D Riccati differential equation to a higher-dimensional Riccati equation. To achieve this, we compare our results with the seven- and eight-order Runge–Kutta method (RK78). It is demonstrated that the continuous solutions via HAM are of comparable accuracy to the discretized approximations obtained by RK78.

2. Basic ideas of HAM

In HAM [8], system (2) is first written in the form,

$$N_i[y_i(t)] = 0, \quad i = 1, 2, \ldots, n,$$

where N_i are a nonlinear operators, t denotes the independent variable and $y_i(t)$ are an unknown function. Note that, not necessary in the nonlinear operator N contain nonlinear term. By means of generalizing the traditional homotopy method, Liao [8] constructs the so-called *zeroth-order deformation equation*

$$(1-q)L[\phi_i(t;q) - y_{i,0}(t)] = qh_iH_i(t)N_i[\phi_i(t;q)], \tag{3}$$

where $q \in [0,1]$ is an embedding parameter, h_i are nonzero auxiliary functions, L is an auxiliary linear operator, $y_{i,0}(t)$ are initial guesses of $y_i(t)$ and $\phi_i(t;q)$ are unknown functions. It is important to note that, one has great freedom to choose auxiliary objects such as h_i and L in HAM. We note that, in the frame of HAM, the solution $y_i(t)$ can be represented by many different base functions such as the polynomial functions, exponential functions, rational functions etc. Obviously, when $q = 0$ and $q = 1$, both

$$\phi_i(t;0) = y_{i,0}(t) \quad \text{and} \quad \phi_i(t;1) = y_i(t),$$

hold. Thus as q increases from 0 to 1, the solution $\phi_i(t;q)$ varies from the initial guess $y_{i,0}(t)$ to the solution $y_i(t)$. Expanding $\phi_i(t;q)$ in Taylor series with respect to q, one has

$$\phi_i(t;q) = y_{i,0}(t) + \sum_{m=1}^{+\infty} y_{i,m}(t)q^m, \tag{4}$$

where

$$y_{i,m}(t) = \frac{1}{m!}\frac{\partial^m \phi_i(t;q)}{\partial q^m}\bigg|_{q=0}. \tag{5}$$

If the auxiliary linear operators, the initial guesses, the auxiliary parameters h_i, and the auxiliary function are so properly chosen, then the series (4) converges at $q = 1$ and

$$\phi_i(t;1) = y_{i,0}(t) + \sum_{m=1}^{+\infty} y_{i,m}(t),$$