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Multistability of neural networks with discontinuous activation function ☆

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Abstract

In this paper, the multistability is studied for two-dimensional neural networks with multilevel activation functions. And it is showed that the system has n^2 isolated equilibrium points which are locally exponentially stable, where the activation function has n segments. Furthermore, evoked by periodic external input, n^2 periodic orbits which are locally exponentially attractive, can be found. And these results are extended to k-neuron networks, which is really enlarge the capacity of the associative memories. Examples and simulation results are used to illustrate the theory.

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Keywords: Neural networks; Multilevel function; Activation function; Multistability

1. Introduction

In the past decades, the studies of neural networks have attracted a tremendous amount of research interest. The dynamical behaviors including stability [1–5], periodic bifurcation and chaos [6–9] of the neural networks have become a focal topic. While the applications of the neural networks range from classifications, associative memory, image processing, and pattern recognition to parallel computation and its ability to solve optimization problems. While the theory on the dynamics of the networks have been developed according to the purposes of the applications.

In some applications, there is a need to design a neural circuit possessing a unique equilibrium point. For example, when solving important classes of optimization problems [10–12], where uniqueness of the equilibrium is required to prevent convergence toward local minima (undesired spurious responses) and hence ensure global optimization. Such a convergent behavior is referred to as ‘‘monostability’’ of a network. Many results on global convergence concern neural networks where the neuron activations are modeled by Lipschitz-continuous

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functions. However, discontinuous neuron activations are of importance and do frequently arise in practice. For example, the classical Hopfield neural networks (HNNs) with graded response neurons [13]. The dynamical behaviors of neural networks with discontinuous activation functions have been studied in [14–16].

On the other hand, when a neural network is employed as an associative memory storage for pattern recognition, the existence of many equilibria is a necessary feature. The notion of "multistability" of a neural network is used to describe coexistence of multiple stable patterns such as equilibria or periodic orbits. The existence of multiple stable patterns has been developed for cellular neural networks in [17–19]. It is found that an k-neuron cellular neural networks can have up to 2^k locally stable equilibria in [20]; and 2^k locally attractive periodic orbits with periodic external inputs in [21]. Some similar results have been found with Hopfield-type neuron activations in [22]. The multistability of neural networks with piecewise linear activation functions has developed in [23,24]. In this paper, we study a type of two-dimensional neural networks with discontinuous neuron activations, which can have n^2 locally stable equilibria, where n is the number of segments of the multilevel activation functions. And n^2 locally attractive periodic orbits can been found with periodic external inputs. In extension, there could been n^k locally stable equilibria in a k-neuron networks. Compared with the previous result [20–24], by using multilevel activation function, we can design neural networks with arbitrary number of stable equilibria which is really enlarge the capacity of associative memories.

The remaining part of this paper is organized as follows. In Section 2 the model and the activation function are given. In Section 3, the number of equilibria of neural networks are obtained. In Section 4, three illustrative examples are provided with simulation results. Finally, conclusions are given in Section 5.

2. Model description

Consider two-dimensional (2-D) neural networks described by the following of differential equations:

$$\begin{cases} \frac{dx_1(t)}{dt} = -x_1(t) + a_{11}f(x_1(t)) + a_{12}f(x_2(t)) + I_1, \\ \frac{dx_2(t)}{dt} = -x_2(t) + a_{21}f(x_1(t)) + a_{22}f(x_2(t)) + I_2 \end{cases} \tag{1}$$

or its equivalent vector form

$$\frac{dx(t)}{dt} = -x(t) + Af(x(t)) + I,$$

where x_i denotes the activity neuron i, $x = (x_1, x_2)^T \in \mathbb{R}^2$ denotes neuron state, $f(x) = (f(x_1), f(x_2))^T$ denotes activation function, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$ is a matrix whose entries represent the synaptic neuron interconnections, and $I = (I_1(t), I_2(t))^T \in \mathbb{R}^2$ is a vector of constant external neuron inputs. If the inputs are ω-periodic, then the neural networks can be written as follows:

$$\begin{cases} \frac{dx_1(t)}{dt} = -x_1(t) + a_{11}f(x_1(t)) + a_{12}f(x_2(t)) + I_1(t), \\ \frac{dx_2(t)}{dt} = -x_2(t) + a_{21}f(x_1(t)) + a_{22}f(x_2(t)) + I_2(t), \end{cases} \tag{2}$$

where the inputs $I(t) = (I_1(t), I_2(t))^T \in \mathbb{R}^2$ is a vector with ω-period.

In the neural networks, Eqs. (1) and (2), the activation function is discontinuous, which has n segments. Choose two arrays of number $\{b_0, b_1, b_2, \ldots, b_n\}$, $\{c_1, c_2, \ldots, c_n\}$ as

$$-1 = c_1 < b_1 < c_2 < b_2 < c_3 < \ldots < b_{n-2} < c_{n-1} < b_{n-1} < c_n = 1; \quad \text{and} \quad b_0 = -\infty, b_n = +\infty.$$

$$f(x) = \begin{cases} c_1, & x < b_1; \\ c_i, & b_{i-1} \leqslant x < b_i; \\ c_n, & x \geqslant b_{n-1}. \end{cases} \tag{3}$$

for $i = 1, 2, \ldots, n$; while $x < b_1 \iff b_0 \leqslant x < b_1$, and $x \geqslant b_{n-1} \iff b_{n-1} \leqslant x < b_n$. So f can be rewritten as:

$$f(x) = c_i, \quad \text{if } b_{i-1} \leqslant x < b_i.$$