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Short communication

About fractional quantization and fractional variational principles

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ABSTRACT

In this paper, a new method of finding the fractional Euler–Lagrange equations within Caputo derivative is proposed by making use of the fractional generalization of the classical *Faá* di Bruno formula. The fractional Euler–Lagrange and the fractional Hamilton equations are obtained within the 1 + 1 field formalism. One illustrative example is analyzed.

1. Introduction

Fractional calculus [1–3], is an emerging field and especially during the last decades it is an alternative tool to solve several complex problems from various fields [4–7].

A fractional derivative represents an operator which generalizes the ordinary derivative. Several definitions of fractional derivatives exist and they were applied successfully in various fields [1–3].

Recently, the fractional variational principles gained importance in studying the fractional mechanics and various versions are investigated. The fractional Lagrangian are constructed from the classical Lagrangian by replacing the classical derivatives with one chosen fractional derivatives and the fractional Euler–Lagrange equations are obtained as a result of a fractional variational procedure [8–15]. During the last years the fractional variational principles have developed and applied to fractional optimal control problems [7,6]. Some type of functional involving the fractional derivatives are started to be used in mathematical economy as well as utilized for describing the dissipative structures arising in nonlinear dynamical systems. Many laws of nature are obtainable by using certain functionals and the classical theory of calculus of variations. However, almost all systems containing internal damping are not suitable to be described properly by the classical methods. The fractional calculus represents one of the promising tools to incorporate in a single theory both conservative and nonconservative phenomena as well.

The main aim of this paper is to obtain the fractional Euler–Lagrange and Hamilton equations within Caputo derivative.

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The plan of the paper is as follows:

In Section 1 some basic definitions regarding the fractional derivatives are presented. In Section 2 the 1 + 1 field formalism is presented. One illustrative example is detailed analyzed in Section 3. Finally, Section 4 is dedicated to the conclusions.

2. Basic tools

In this section, we formulate the problem in terms of the left and the right Riemann–Liouville (RL) fractional derivatives and the Caputo derivative. The left RL is defined as

$$ {}_a\mathbf{D}_t^\alpha f(t) = \frac{\left(\frac{d}{dt}\right)^n}{\Gamma(n-\alpha)} \int_a^t (-\tau + t)^{n-\alpha-1} f(\tau) d\tau, \tag{1} $$

and the right RL fractional derivative has the form

$$ {}_t\mathbf{D}_b^\alpha f(t) = \frac{\left(-\frac{d}{dt}\right)^n}{\Gamma(n-\alpha)} \int_t^b (\tau - t)^{n-\alpha-1} f(\tau) d\tau. \tag{2} $$

Here, the order α fulfills $n-1 \leqslant \alpha < n$ and Γ represents the Euler's Gamma function. By direct calculation we observe that the RL derivative of a constant is not zero, namely ${}_a\mathbf{D}_t^\alpha C = C\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}$.

The corresponding Caputo's fractional derivatives are defined as follows: *the left Caputo Fractional Derivative*

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n f(\tau) d\tau, \tag{3} $$

and *the right Caputo Fractional Derivative*

$$ {}_t^C D_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (\tau - t)^{n-\alpha-1} \left(-\frac{d}{d\tau}\right)^n f(\tau) d\tau, \tag{4} $$

The Caputo derivative of a constant is zero. For $0 < \alpha < 1$ we have

$$ {}_a\mathbf{D}_t^\alpha f(t) = {}_a^C D_t^\alpha f(t) - \frac{(t-a)^\alpha f(a)}{\Gamma(1-\alpha)}, \tag{5} $$

and

$$ {}_a\mathbf{D}_t^\alpha \phi(t) = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} \phi(t) + \sum_{k=1}^{\infty} \binom{\alpha}{k} \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} \phi^{(k)}(t), \tag{6} $$

under the assumption $t > a$ [1]. We mention that the above formula is obtained by using the fractional derivative of the Heaviside unit-step function (see for more details [1]).

3. Non-local Lagrangians within 1 + 1 field theories

In the following we present briefly a new formalism of fractional Hamiltonian formulation.

The first step is to consider that the dynamical variable $q(t)$ is a 1 + 1 dimensional field $Q(x,t)$ subjected to the following chirality condition [16]:

$$ \frac{dQ(x,t)}{dt} = \partial_x Q(x,t). \tag{7} $$

A feature of the Hamiltonian formalism for non-local theories is that is contains the Euler–Lagrange equations as Hamiltonian constraints. By making use of (7) we get

$$ (\frac{d}{dt})^n Q(x,t) = (\partial_x)^n Q(x,t), n \in N_0, \tag{8} $$

having in mind that $Q(x,t) = q(x+t)$ assures the one-to-one correspondence between $q(t)$ and $Q(x,t)$ [16]. Ostrogradski's coordinates are defined as in the following:

$$ Q^{(n)}(t) = (\partial_x)^n Q(x,t)|_{x=x_0}, \tag{9} $$

where the discontinuity curve $x_0(t) = x_0$ is a constant [16]. By using the inverse relation provided by the Taylor expansion around $x = x_0$ we obtain

$$ Q(x,t) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} Q^{(n)}(t). \tag{10} $$