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Short communication

Analytical method based on Walsh function combined with orthogonal polynomial for fractional transport equation

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ABSTRACT

In this paper a novel method based on Walsh function combined with a Chebyshev polynomials of the first kind was applied for the resolution of fractional transport equation in three-dimensions. A specific application of the method is discussed.

1. Introduction

The spectral methods establish an analytical formulation to be able to find exact solutions for approximations of the transport equation.

In the ordinary cases several approaches have been suggested among them we mention the method proposed by Chandrasekhar [1] solves analytically the discrete equations. This method which is called S_N the *SGF* method [2,3] is a numerical nodal method that generates a numerical solution for the S_N equations in slab geometry that is completely free of spatial truncation error. The LTS_N method [4] solve analytically the S_N equations employing the Laplace transform technique in the spatial variable. Recently, following the idea encompassed by the LTS_N method we have derived a generic method prevailing the analyticity for solving one-dimensional approximation that transform the transport equation into a set of differential equations.

Fractional transport equation is an interesting issue to be analyzed mainly because it can describe better the anomalous processes. In our recent work we have presented a new approximation, namely the one dimensional fraction integro-differential equation was converted into a system of fractional differential equation based on the use of the Chebyshev polynomials [5].

There are three classes of set of orthogonal functions which are widely used. The first includes sets of piecewise constant basis functions (e.g. Walsh, block-pulse). The second consists of sets of orthogonal polynomials (e.g. Laguerre, Legendre, Chebyshev). The third is the widely used sets of sine–cosine functions in Fourier series.

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In this paper the Walsh function is used for solving the three-dimensional case of fractional transport equation. This method is based on expansion of the angular flux in a truncated series of Walsh function in the angular variable. By replacing this development in the transport equation, it will result a first-order fractional linear differential system.

The paper is organized as follows.

Section 2 contains a definition and some properties of Walsh function. In Section 3 we present some basic results and definitions of fractional derivatives. Section 4 describes how to convert a fractional transport equation into a first-order fractional linear differential equation system by making use of the Chebyshev polynomials. In Section 5 we report the convergence of the spectral solution. Finally, we give a specific application of this method in Section 6.

2. Walsh function

The Walsh function has many properties similar to those of the trigonometric functions. For example it forms a complete, total collection of functions with respect to the space Lebesgue functions. However, these functions are simpler in structure that the trigonometric functions namely because they take only the values 1 and −1. As a result, they may be expressed as linear combinations of the Haar functions [6], so many proofs about the Haar functions carry over to the Walsh system easily. Moreover, the Walsh functions are Haar wavelets packets [7]. We use the ordering of the Walsh functions due to Paley [8]. Any function $f \in L^2[0,1]$ can be expanded as a series of Walsh functions

$$f(x) = \sum_{i=0}^{\infty} c_i W_i(x) \quad \text{where} \quad c_i = \int_0^1 f(x) W_i(x). \tag{1}$$

Fine [9] has discovered an important property of the Walsh Fourier series, namely the $m = 2^n$ th partial sum of the Walsh series of a function f is piece-wise constant, equals to the L^1 mean of f, on each subinterval $[(i-1)/m, i/m]$. As a result in applications the Walsh series are always truncated to $m = 2^n$ terms. The corresponding coefficients c_i of the Walsh (-Fourier) series are given below

$$c_i = \sum_{j=0}^{m-1} \frac{1}{m} W_{ij} f_j, \tag{2}$$

where f_j is the average value of the function $f(x)$ in the jth interval of width $1/m$ in the interval $(0,1)$, and W_{ij} represents the value of the ith Walsh function in the jth subinterval. The order m Walsh matrix, $\mathscr{W}$, has elements W_{ij}.

Let $f(x)$ have Walsh series with coefficients c_i and its integral from 0 to x have a Walsh series with coefficients $b_i : \int_0^x f(t)dt = \sum_{i=0}^{\infty} b_i W_i(x)$. If we truncate to $m = 2^n$ terms and use the obvious vector notation, then integration is performed by matrix multiplication $\mathbf{b} = P_m^T \mathbf{c}$, where

$$P_m^T = \begin{bmatrix} P_{m/2} & \frac{1}{2m} I_{m/2} \\ -\frac{1}{2m} I_{m/2} & O_{m/2} \end{bmatrix}, \quad P_m^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{4} & 0 \end{bmatrix}, \tag{3}$$

and I_m is the unit matrix, O_m is the zero matrix (of order m), [10].

3. Fractional calculus tools

Before starting to study the three-dimensional spectral solution of a fractional transport equation we enlist some definitions and basic results regarding the fractional calculus (see for more details Refs. [16–23]).

A real function $f(x), x > 0$ is said to be in the space $C_{\alpha,\alpha \in R}$ if there exists a real number $p(> \alpha)$, such that $f(x) = x^p f_1(x)$ where $f_1(x) = C[0,\infty)$. Clearly $C_\alpha \subset C_\beta$ if $\beta \leqslant \alpha$.

A function $f(x), x > 0$ is said to be in space $C_\alpha^m, m \in N \cup \{0\}$, if $f_{(m)} \in C_\alpha$.

The left sided Riemann-Liouville fractional integral of order $\mu > 0$, of a function $f \in C_\alpha, \alpha \geqslant 1$ is defined as:

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau, \quad \mu > 0, \quad t > 0, \tag{4}$$

$$I^0 f(t) = f(t). \tag{5}$$

The left sided Riemann-Liouville fractional derivative of $f, f \in C_{-1}^m, m \in N \cup \{0\}$ of order $\alpha > 0$, is defined as:

$$D^\mu f(t) = \frac{d^m}{dt^m} I^{m-\mu} f(t), \quad m-1 < \mu \leqslant m, \quad m \in N. \tag{6}$$

The left sided Caputo fractional derivative of $f, f \in C_{-1}^m, m \in N \cup \{0\}$ of order $\alpha > 0$, is defined as:

$$D_c^\mu f(t) = \begin{cases} \left[I^{m-\mu} f^{(m)}(t)\right], m-1 < \mu \leqslant m, \quad m \in N, \\ \frac{d^m}{dt^m} f(t) \mu = m. \end{cases} \tag{7}$$

Note that we have the following properties