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Review

Cycle expansions: From maps to turbulence

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ABSTRACT

We present a derivation, a physical explanation and applications of cycle expansions in different dynamical systems, ranging from simple one-dimensional maps to turbulence in fluids. Cycle expansion is a newly devised powerful tool for computing averages of physical observables in nonlinear chaotic systems which combines many innovative ideas developed in dynamical systems, such as hyperbolicity, invariant manifolds, symbolic dynamics, measure theory and thermodynamic formalism. The concept of cycle expansion has a deep root in theoretical physics, bearing a close analogy to cumulant expansion in statistical physics and effective action functional in quantum field theory, the essence of which is to represent a physical system in a hierarchical way by utilizing certain multiplicative structures such that the dominant parts of physical observables are captured by compact, maneuverable objects while minor detailed variations are described by objects with a larger space and time scale. The technique has been successfully applied to many low-dimensional dynamical systems and much effort has recently been made to extend its use to spatially extended systems. For one-dimensional systems such as the Kuramoto–Sivashinsky equation, the method turns out to be very effective while for more complex real-world systems including the Navier–Stokes equation, the method is only starting to yield its first fruits and much more work is needed to enable practical computations. However, the experience and knowledge accumulated so far is already very useful to a large set of research problems. Several such applications are briefly described in what follows. As more research effort is devoted to the study of complex dynamics of nonlinear systems, cycle expansion will undergo a fast development and find wide applications.

Contents

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1. Introduction

In a chaotic system, a small initial uncertainty is exponentially magnified with time [98]. Due to this initial uncertainty a specific orbit can only be followed for a short time, so individual events cannot be predicted with precision in the long time run. Nevertheless, due to ergodicity of the chaotic motion, the average behavior over a swarm of initial conditions or over a long time is still predictable. In our physical observations, often only such averages are measured.

In chaotic motion the long-time evolution of any "typical" initial conditions leads to the same asymptotic distribution – the so-called invariant measure, which is invariant under the given dynamics and conveniently used in computing averages of physical observables without resorting to the long-time evolution.

Periodic solutions (also called cycles in this review) make closed curves and quasi-periodic orbits move on tori in the phase space. However, such nice, simple invariant sets are exceptions rather than rules. In chaotic systems, the asymptotic invariant geometric objects have extremely complex structures with fractal dimensions and thus are often called "strange attractors". Invariant measures inevitably become singular on these sets. Since the strange attractor is densely covered by unstable periodic orbits (UPOs) invariant measures are fully captured by them, which has been well addressed in the periodic orbit theory [22,4,26,8].

Practical computations become much more efficient after cycle expansion technique is introduced. Cycle expansion utilizes short periodic orbits to capture the major portion of the invariant measure, longer cycles being used to deliver systematic curvature corrections. Thus, cycle expansion naturally brings about a hierarchical structure on the set of all periodic orbits. As we shall see in Section 2, the cycle order is usually made according to the topological length of an orbit, but other orderings are also possible, such as the stability ordering. With finite resolution, we only need to compute finitely many periodic orbits for physical averages on a hyberbolic set. Short periodic orbits are relatively easy to obtain since they are finite closed orbits and the exponential growth of uncertainty has yet to effect.

Cycle expansion was first proposed by Cvitanović et al. [4,26,25] and was later explored and utilized by many researchers [22]. A rather complete account could be found in the monograph "*Chaos: quantum and classical*" [22]. In other areas of physics, similar ideas have been used for long time, e.g., the cumulant expansion in statistical physics [85]. In all these methods, certain types of multiplicative structures in the characteristic equation of a physical system are preserved and utilized through a perturbation expansion on the exponentials, which results in much faster convergence. The main philosophy is to approximate the system under investigation by easy-maneuverable finite solutions and patterns the repetition of which captures the dominant part of higher-order terms, leaving only exponentially small corrections. This property is termed shadowing in dynamical systems theory [108,32]: a long chaotic orbit could be approximated by pieces of shorter orbits. But the concept is certainly generalizable to many other physical models. So, in a broader sense, cycle expansion is a general tool for computing averages in many nonlinear physical systems. In this review, we mainly focus its application on dynamical systems but other applications will also be explained briefly.

Progress in the study of nonlinear dynamical systems has revealed universality of their behavior [38,19]. For example, one-dimensional map [35] and rotating fluid [19] may experience similar route to chaos. So, concepts and methods developed in one system may be well applicable to other similar systems. Early studies of cycle expansion are mainly concentrated on low-dimensional maps and ordinary differential equations (ODEs) and have been very successful in disclosing their properties and computing physical observables in different situations [22]. Recent developments move towards its application in spatially extended systems described by nonlinear partial differential equations (PDEs) such as the complex Ginzburg–Landau equation (CGLe) [3,114,69,71], the Kuramoto–Sivashinsky equation (KSe) [65,102,68] and the