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Sawtooth disruptions and limit cycle oscillations

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Abstract

A minimal (low-dimensional) dynamical model of the sawtooth oscillations is presented. It is assumed that the sawtooth is triggered by a thermal instability which causes the plasma temperature in the central part of the plasma to drop suddenly, leading to the sawtooth crash. It is shown that this model possesses an isolated limit cycle which exhibits relaxation oscillation, in the appropriate parameter regime, which is the typical characteristics of sawtooth oscillations. It is further shown that the invariant manifold of the model is actually the slow manifold of the relaxation oscillation.

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1. Introduction

Sawtooth oscillations [1], commonly observed in current carrying, magnetically confined plasmas, are believed to be the result of resistive internal kink mode, i.e., $(m = 1, n = 1)$ oscillation. These oscillations are characterized by a relatively slow rise of the electron temperature in the central region of the plasma column followed by a rapid drop (the crash). This is a typical nature of a *stick-slip* or *relaxation* oscillation [2], where the stress is slowly built up and then suddenly released after a certain threshold, observed in many other dynamical systems, e.g., Portevin–Le Chatelier effect [3], a matter of interest in material science.

In this work, we propose a *minimal* (low-dimensional) dynamical model for sawtooth oscillation in tokamaks, based on a transport catastrophe due to a thermal instability [4]. Low-dimensional or low-order dynamical system, i.e., a system of coupled ordinary differential equations, has several advantages over a detailed physical model of the actual system in the understanding of the global behaviour of the system. These models become powerful as they are supported by well developed mathematical theories which can be used to gain insight into the qualitative behaviour of the system such as bifurcation and stability [5]. Several examples of these models can be found in the context of understanding the behaviour of fusion plasmas ranging from the edge localized modes (ELM) [6] to plasma turbulence [7].

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In spite of a great deal of experimental and theoretical research, the sawteeth in tokamaks continue to be a subject of exploration. For example, though numerical simulations [8] and some experimental results [9] suggest total reconnection within the safety factor $q = 1$ surface during a sawtooth crash, there are also experimental evidences [10], which indicate that q remains well below unity during a sawtooth cycle. Apparently, there have been several important contributions to the understanding of the sawtooth dynamics, e.g., sawteeth with partial reconnection, based on turbulent transport [13]. A Taylor relaxation model of the sawteeth has also been considered by Gimblett and Hastie [14].

Here, we primarily focus on the dynamics of sawtooth oscillations based on a low-dimensional dynamical model. We rigorously prove that this dynamical model of the sawteeth based on thermal instability, besides capturing the important physical aspects, does exhibit well defined, isolated limit cycle oscillations, characteristics of self-excited relaxation phenomena like the sawteeth. Alternatively, several authors have proposed Hamiltonian models [15–17], which however can have infinite number of periodic solutions depending on the starting points of the evolution with the same set of physical parameters, which is rather inconsistent with the universal nature of sawteeth for similar experimental conditions. These models, despite having dissipation, possess a conserved quantity much like the Hamiltonian of a conjugate system.

In Section 2, we formulate the minimal dynamical model. In Section 3, we analyse the bifurcation and stability of the system, pointing out the existence of an isolated and unique limit cycle and its global stability. We complete this section by proving the uniqueness of the limit cycle where we demonstrate the existence of an algebraic equation for the limit cycle. Next, in Section 4, we address the issue of relaxation oscillation and explore the parameter regime where the limit cycle exhibits sawtooth-like oscillations. In Section 5, with the help of a renormalization group method, we prove that the invariant manifold of the dynamical system is indeed the slow manifold of the relaxation oscillation. In the Appendix, we outline the Hamiltonian approach to this dynamical model.

2. Dynamical modeling of sawtooth oscillations

Typical sawtooth oscillations in small tokamaks ($R_0 = 1$ m) exhibit linear growth of central temperature with few milliseconds of duration and rapid crash time of $\sim$ several microseconds, in the Ohmic heating phase [18,19,10]. Although there are several other exotic cases, viz., giant and monster sawteeth [11,12], we shall limit our discussion to the simpler type of sawteeth with a linear rise of the electron temperature. The dynamical system, controlling the sawtooth oscillations of the central electron temperature, then can be written as [4,15,20]

$$\frac{3}{2}n\frac{\partial T_{\rm e}}{\partial t} = E_{\|}^2\sigma_{\|} - \nu_L(A,T_{\rm e})nT_{\rm e}, \tag{1}$$

$$\frac{\partial A}{\partial t} = \gamma(T_{\rm e})A, \tag{2}$$

where $T_{\rm e}$ is the central electron temperature expressed in energy units, A is the amplitude of the oscillation, $\nu_L(A,T_{\rm e})$ is the rate of temperature redistribution, and $\gamma(T_{\rm e})$ is the growth rate of the relevant mode. The collisional parallel conductivity $\sigma_{\|} \propto T_{\rm e}^{3/2}$. In the above equations, the particle density n remains nearly constant during the sawtooth cycle, which is consistent with experimental observations. We further note that in the cases of sawtooth oscillations, we are going to consider, the classical diffusion time [21] for the plasma current within the $q \leqslant 1$ volume, $\tau_J = (r_1/d_{\rm e})^2/\nu_{\rm ei}$ (where r_1 is the radius of the $q = 1$ surface, $d_{\rm e}$ is the plasma skin depth, and $\nu_{\rm ei}$ is the electron–ion collision frequency), is one order of magnitude higher than the sawtooth repetition time. For example, in the Alcator C-Mod (MIT) machine, typically, $\tau_J \sim$ 80–400 ms for Ohmic regimes [22], whereas the sawtooth period (crash time being negligibly smaller than the period) $\tau_{\rm st} \sim 4$ ms. Therefore it can be safely assumed that the current redistribution does not play any significant role and the corresponding $E_{\|}$ remains constant. We further assume that the pressure redistribution parameter $\nu_L(A,T_{\rm e})$ can be expressed with simple power laws, i.e., $\nu_L(A,T_{\rm e}) \propto T_{\rm e}^{\alpha}A^{\sigma}$, where α and σ are arbitrary constants.

With these considerations, we note that the second term in Eq. (1) is responsible for the sawtooth crash. However, in absence of this term the general solution of Eq. (1) is explosive. In particular, it should include