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Universal theory of dynamical chaos in nonlinear dissipative systems of differential equations

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Abstract

A new universal theory of dynamical chaos in nonlinear dissipative systems of differential equations including ordinary and partial, autonomous and non-autonomous differential equations and differential equations with delay arguments is presented in this paper. Four corner-stones lie in the foundation of this theory: the Feigenbaum's theory of period doubling bifurcations in one-dimensional mappings, the Sharkovskii's theory of bifurcations of cycles of an arbitrary period up to the cycle of period three in one-dimensional mappings, the Magnitskii's theory of rotor type singular points of two-dimensional non-autonomous systems of differential equations as a bridge between one-dimensional mappings and differential equations and the theory of homoclinic cascade of bifurcations of stable cycles in nonlinear differential equations. All propositions of the theory are strictly proved and illustrated by numerous analytical and computing examples.

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1. Introduction

Present-day classical conception of dynamical chaos in dissipative systems of differential equations has the next main features [1–10]:

(a) numerous definitions of irregular attractors: strange, chaotic, stochastic, quasi, Lorenz type attractor and so on;
(b) full misunderstanding of nature of irregular attractors in terms of trajectories of differential equations;
(c) existence of complex irregular attractor only in a neighbourhood of a saddle-node or a saddle-focus singular point of autonomous system (homoclinic chaos);

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(d) hypothesis on exponential divergence of trajectories inside an irregular attractor, assumption that Lyapunov exponents should satisfy the condition $\lambda_1 < 0$, $\lambda_2 = 0$, $\lambda_3 > 0$ in the three-dimensional case;
(e) geometrical approach to analysis of nonlinear differential equations – reduction to a Poincare map and attempt to prove an existence of its invariant fractal set in a neighbourhood of a singular point as a some kind of the Smale's horseshoe;
(f) hypothesis on fractal structure of irregular attractor and assumption that its fractal dimension lies in the interval $2 < d_F < 3$ in the three-dimensional case;
(g) assumption that there exist three scenarios of transition to chaos in dissipative systems of ordinary differential equations – Feigenbaum's cascade of period doubling bifurcations of stable cycles [11], Ruelle–Takens scenario of transition to chaos through a destruction of three-dimensional torus [12], and transition to chaos through an intermittency that was discovered by Pomeau and Manneville [13];
(h) Ruelle–Takens hypothesis of transition to chaos in partial nonlinear differential equations through a destruction of three-dimensional torus.

In the present paper, it is offered a new universal theory of dynamical chaos in nonlinear dissipative systems of differential equations including ordinary and partial, autonomous and non-autonomous differential equations and differential equations with delay arguments. This new theory has only single coincidence with the present-day classical theory of dynamical chaos, that is, the Feigenbaum's cascade of bifurcations. The new theory has the next main features in comparison with the features of the present-day theory:

(a) existence of only single type of irregular attractors in all dissipative systems of ordinary and partial differential equations – singular attractors;
(b) clear understanding of nature of singular attractors in terms of trajectories of differential equations;
(c) statement that existence of a singular cycle, but not a singular point is necessary condition of generation of singular attractors in autonomous systems of ordinary differential equations;
(d) statement that phase shift of trajectories, but not their exponential divergence is in actual fact a nature of chaotic dynamics in dissipative systems of differential equations, Lyapunov exponents of any singular attractor of a three-dimensional autonomous system should satisfy the condition $\lambda_1 < 0$, $\lambda_2 = 0$, $\lambda_3 = 0$;
(e) construction of some revolving section transversally to the singular cycle and transition to analysis of a new system of nonlinear differential equations of lower dimension in this section, but not in the Poincare section that results to the phase loss;
(f) in three-dimensional case, all regular (stable limit cycles) and singular attractors and unstable cycles belong to the two-dimensional surface that is a closure of the two-dimensional invariant unstable manifold of a singular saddle cycle, it means that fractal dimension of any singular attractor is not more then two;
(g) there is only one universal scenario of transition to chaos and generation of singular attractors in nonlinear dissipative systems of ordinary differential equations – the Feigenbaum's cascade of period doubling bifurcations of original singular saddle cycle, then the Sharkovskii's subharmonic cascade of bifurcations of stable cycles with arbitrary period up to the cycle of period three [14], then the homoclinic cascade of bifurcations of stable cycles converging to a homoclinic contour of a singular point discovered by Magnitskii in [15], then more complex continuation of this scenario is possible;
(h) transition to chaos in partial nonlinear differential equations through a subharmonic cascade of bifurcations of two-dimensional tori, discovered in [16,17].

Four remarkable results lie in the foundation of this new theory: the Feigenbaum's theory of period doubling bifurcations in one-dimensional mappings, the Sharkovskii's theory of bifurcations of cycles of an arbitrary period up to the cycle of period three in one-dimensional mappings, the Magnitskii's theory of rotor type singular points of two-dimensional non-autonomous systems of differential equations [18–20] as a bridge between one-dimensional mappings and differential equations and the theory of homoclinic cascade of bifurcations of stable cycles in nonlinear differential equations. All propositions of the theory are strictly and in detail proved in [11,14,18–21]. In the present paper we state briefly the main results of this Feigenbaum–Sharkovskii–Magnitskii theory and show with numerous analytical and computing examples that all classical