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Analytic study on the generalized fifth-order KdV equation: New solitons and periodic solutions

Abdul-Majid Wazwaz *

Department of Mathematics and Computer Science, Saint Xavier University, 3700 W. 103rd Street, Chicago, IL 60655, United States

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Abstract

An analytic study is conducted on a generalized fifth-order KdV equation. The tanh method and a sinh–cosh functions ansatz are used. A set of entirely new solitons and periodic solutions is established. The study introduces new ansatz to handle nonlinear PDEs in the solitary wave theory.

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1. Introduction

Scientific fields, such as solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics are usually governed by nonlinear PDEs [1–13]. A broad range of analytical solution methods, such as inverse scattering method [4], Bucklund transformation method [6,7], Hirota's bilinear scheme, pseudo spectral method, Jacobi elliptic method, Painlevé analysis, the tanh method [14–16], and other methods are used in the analysis of these problems. Moreover, several numerical methods, such as the finite differences method, collocation method and Galerkin method, were employed for numerical treatment of the nonlinear problems. However, some of these analytical and numerical solutions methods are not easy to use and sometimes require from tedious works and calculations.

This paper is concerned with a generalized fifth-order KdV equation (gfKdV) of the form

$$u_t + au^n u_x + bu^n u_{3x} + u_{5x} = 0, \quad n \geqslant 1, \tag{1}$$

with constant parameters a and b and $u_{kx} = \frac{\partial^k u}{\partial x^k}$.

The generalized KdV equations describe motions of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice [1–7]. The nonlinear generalized KdV equation is an important

* Tel.: +1 773 298 3397; fax: +1 773 779 9061.
E-mail address: wazwaz@sxu.edu

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mathematical model with wide applications in quantum mechanics and nonlinear optics. Typical examples are widely used in various fields such as solid state physics, plasma physics, fluid physics and quantum field theory.

It is the purpose of this paper to ascertain what effect dispersion and the exponent n have on travelling wave solutions. The objectives of this work are twofold. Firstly, we seek to employ the well known tanh method and a sinh–cosh ansatz to establish exact solutions for (1) for all values of $n \geqslant 1$. Secondly, we aim to emphasize the applicability of these two schemes in providing useful guidance for other related nonlinear problems.

Two distinct approaches will be used, namely the tanh method [13–16] and the sinh–cosh ansatz [8]. The schemes that will be used have the advantage of reducing the nonlinear problem to a system of algebraic equations that can be easily solved by using symbolic computation such as *Mathematica* or *Maple*. In what follows, we highlight the main features of the proposed methods. The power of the methods, that will be used, is its ease of use to determine shock or solitary type of solutions.

2. The methods

We first unite the independent variables x and t into one wave variable $\xi = x - ct$ to carry out a PDE in two independent variables

$$P(u, u_t, u_x, u_{xx}, u_{xxx}, \ldots) = 0, \tag{2}$$

into an ODE

$$Q(u, u', u'', u''', \ldots) = 0. \tag{3}$$

Eq. (3) is then integrated as long as all terms contain derivatives. Usually the integration constants are considered to be zeros in view of the localized solutions. However, the nonzero constants can be used and handled as well.

2.1. The sinh–cosh ansatz

It is useful to apply the new sinh–cosh ansatz introduced in [8]. This approach was formally proved to provide new travelling wave solutions. A specific ansatz of the form

$$u(x,t) = \left(\frac{\alpha}{1 + \lambda f(\mu\xi)}\right)^{\frac{1}{n}}, \quad \xi = x - ct, \tag{4}$$

is introduced to obtain more solitons solutions, where α, λ and μ are parameters that will be determined, and $f(\mu\xi)$ takes the hyperbolic cosine or the hyperbolic sine functions. As will be seen later, the cosh ansatz provides solitons solutions, whereas the sinh ansatz mostly provides complex solutions. The approach is simply used by applying (4) into Eq. (1), collecting the coefficients of the resulting hyperbolic functions and setting it to zero, and solving the resulting equations to determine the parameters α, λ, and μ.

2.2. The tanh method

The main features of the tanh method will be reviewed briefly because details can be found in [14,15] and the references therein. The standard tanh method introduces the tanh function as a new variable, since all derivatives of a tanh are represented by a tanh itself. The tanh method introduces a new independent variable

$$Y = \tanh(\mu\xi), \tag{5}$$

that leads to the change of derivatives:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi} &= \mu(1 - Y^2)\frac{\mathrm{d}}{\mathrm{d}Y}, \\ \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} &= \mu^2(1 - Y^2)\left(-2Y\frac{\mathrm{d}}{\mathrm{d}Y} + (1 - Y^2)\frac{\mathrm{d}^2}{\mathrm{d}Y^2}\right). \end{aligned} \tag{6}$$