Commun Nonlinear Sci Numer Simulat 14 (2009) 1152–1156

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Analysis of nonlinear fractional partial differential equations with the homotopy analysis method

Hang Xu *, Shi-Jun Liao, Xiang-Cheng You

School of Naval Architecture, Ocean and Civil Engineering, Shanghai Jiao Tong University, No. 1954 Huashan Road, Shanghai 200030, China

ARTICLE INFO

Article history:
Received 15 April 2008
Received in revised form 15 April 2008
Accepted 16 April 2008
Available online 25 April 2008

PACS:
02.30.Mv
05.45.Df

Keywords:
Series solution
Fractional
Caputo fractional derivative

ABSTRACT

In this paper, the time fractional partial differential equations are investigated by means of the homotopy analysis method. This technique is extended to study the partial differential equations of fractal order for the first time. The accurate series solutions are obtained. This indicates the validity and great potential of the homotopy analysis method for solving nonlinear fractional partial differential equations.

1. Introduction

The idea of derivatives of noninteger order was initially appeared in a letter from Leibnizs to L'Hospital in 1695. For three centuries, studies of the theory of fractional order were mainly constraint to the field of pure theoretical mathematics, which were only useful for mathematicians. In the last several decades, many researchers found that derivatives of noninteger order are very suitable for the description of various physical phenomena such as rheology, damping laws, diffusion process. These findings invoked the growing interest of studies of the fractal calculus in various fields such as physics, chemistry and engineering. Several excellent books and papers describing the state-of-the-art available in the literature testify to the maturity of theory of fractal order. Podlubny [1] provided the solutions method of differential equations of arbitrary real order and applications of the described methods in various fields. Oldham and Spanier [2] gave a systematic presentation of the ideas, methods, and applications of the fractional calculus. Their book played an important role in the development of the theory of fractional order. Some other fundamental works on various aspects of the fractional calculus are given by Gorenflo and Vessella [3], Kiryakova [4], Miller and Ross [5], Rossikhin and Shitikova [6], etc.

Fractal differential equations have attracted many researchers [7–10] due to their important applications in science and engineering such as modelling of anomalous diffusive and sub-diffusive systems, description of fractional random walk, unification of diffusion and wave propagation phenomena. This type of equations are obtained from the classical diffusion or wave equation by replacing the first- or second-order time derivatives term by a fractional derivative of order $\alpha > 0$.

* Corresponding author. Tel.: +86 21 62932356; fax: +86 21 62933156.
E-mail addresses: hangxu@sjtu.edu.cn (H. Xu), sjliao@sjtu.edu.cn (S.-J. Liao), xcyou@sjtu.edu.cn (X.-C. You).

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doi:10.1016/j.cnsns.2008.04.008

In this paper, the homotopy analysis method [11–20] is applied to solve fractional partial differential equations. A new approach for solving the fractional partial differential equations is established. It is expected the proposed techniques can be further applied to derive solutions for other partial differential equations with fractional order.

The fractional differential equations to be solved of the form

$$\frac{\partial u^\alpha}{\partial t^\alpha} = f(x,y,z)u_{xx} + g(x,y,z)u_{yy} + h(x,y,z)u_{zz}, \tag{1}$$

subject to the Neumann boundary conditions

$$u_x(0,y,z,t) = f_1(y,z,t), \quad u_x(a,y,z,t) = f_2(y,z,t), \tag{2}$$
$$u_y(x,0,z,t) = g_1(x,z,t), \quad u_y(x,b,z,t) = g_2(x,z,t), \tag{3}$$
$$u_z(x,y,0,t) = h_1(x,y,t), \quad u_z(x,y,c,t) = h_2(x,y,t), \tag{4}$$

and the initial conditions

$$u(x,y,z,0) = \psi(x,y,z), \quad u_t(x,y,z,0) = \theta(x,y,z), \tag{5}$$

where α being a constant describing the fractional derivative. When $0 < \alpha \leqslant 1$, Eq. (1) can be reduced to a fractional heat-like equation, and to a wave-like equation for $1 < \alpha \leqslant 2$.

2. Fractional calculus

Several definitions of fractional calculus have been proposed in the last two centuries. Here, we employed the Riemann–Liouville [1] definitions of fractal derivative operator J_α^α.

Definition 1. The Riemann–Liouville integral operator of order α on the usual Lebesque space $L_1[a,b]$ is given by

$$J_\alpha^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_\alpha^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad (\alpha > 0), \tag{6}$$
$$J_\alpha^0 f(x) = f(x). \tag{7}$$

It has the following properties: (i) J_a^α exists for any $x \in [a,b]$, (ii) $J_a^\alpha J_a^\beta = J_a^{\alpha+\beta}$, (iii) $J_a^\alpha J_a^\beta = J_a^\beta J_a^\alpha$, (iiii) $J_a^\alpha (x-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(x-a)^{\alpha+\gamma}$, where $f \in L_1[a,b],\ \alpha,\beta \geqslant 0$ and $\gamma > -1$.

It is worth mentioning that the Riemann–Liouville derivative has certain disadvantages for describing some natural phenomena with fractional differential equations. Thus, we introduce Caputo's definition [21] of fractal derivative operator D^α, which is a modification of Riemann–Liouville definition.

Definition 2. The Caputo's definition [21] of fractal derivative operator is given by

$$D^\alpha f(x) = J^{n-\alpha}D^n f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t)\mathrm{d}t, \tag{8}$$

where $n-1 < \alpha < n,\ n \in \mathbb{N},\ x > 0$. It has the following two basic properties.

Lemma 1. *For $n-1 < \alpha \leqslant n$ and $f \in L_1[a,b]$, it holds*

$$D_a^\alpha J_a^\alpha f(x) = f(x),$$
$$J_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} f^{(\kappa)}(0^+)\frac{x-a^k}{k!}, \quad x > 0.$$

For more mathematical properties of fractional derivatives and integrals, please refer to the related references in this subject.

3. Solution method

In this section, we consider a nonlinear equation in a general form:

$$\mathcal{N}[f(\mathbf{r},t)] = 0, \tag{9}$$

where $\mathcal{N}$ is a nonlinear operator, $f(\mathbf{r},t)$ is an unknown function, and $\mathbf{r}$ and t are independent variables, respectively. Let $f_0(\mathbf{r},t)$ denote an initial approximation of the solution of Eq. (9), $\hbar$ a nonzero auxiliary parameter, $H(\mathbf{r},t)$ a nonzero auxiliary function, and $\mathcal{L} = \mathcal{D}_t^\alpha$ an auxiliary linear operator with the following property:

$$\mathcal{L}[\phi(\mathbf{r},t)] = 0 \quad \text{when } \phi(\mathbf{r},t) = 0. \tag{10}$$

Then we construct the HAM deformation equation in the following form:

$$(1-q)\mathcal{L}[\Phi(\mathbf{r},t;q) - f_0(\mathbf{r},t)] = q\ \hbar\ H(\mathbf{r},t)\ \mathcal{N}[\Phi(\phi(\mathbf{r},t;q))] \tag{11}$$