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On algorithms for estimating computable error bounds for approximate periodic solutions of an autonomous delay differential equation

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Abstract

Machine tool chatter has been characterized as isolated periodic solutions or limit cycles of delay differential equations. Determining the amplitude and frequency of the limit cycle is sometimes crucial to understanding and controlling the stability of machining operations. In Gilsinn [Gilsinn DE. Computable error bounds for approximate periodic solutions of autonomous delay differential equations, Nonlinear Dyn 2007;50:73–92] a result was proven that says that, given an approximate periodic solution and frequency of an autonomous delay differential equation that satisfies a certain non-criticality condition, there is an exact periodic solution and frequency in a computable neighborhood of the approximate solution and frequency. The proof required the estimation of a number of parameters and the verification of three inequalities. In this paper the details of the algorithms will be given for estimating the parameters required to verify the inequalities and to compute the final approximation errors. An application will be given to a Van der Pol oscillator with delay in the nonlinear terms.
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1. Introduction

Machine tool dynamics has been modeled using delay differential equations for a number of years as is clear from the vast literature associated with it. For a detailed review of machining dynamics see Tlusty [1]. For a discussion of dynamics in milling operations see Balchandran [2] and Zhao and Balachandran [3]. For drilling operations see Stone and Askari [4] and Stone and Campbell [5]. For an analysis of chatter occurring in

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turning operations see Hanna and Tobias [7], Marsh et al. [8], and Nayfeh et al. [9]. Machine tool chatter is undesirable self-exited periodic oscillations during machining operations. It has been identified as a Hopf bifurcation of limit cycles from steady state solutions. For a way of estimating the critical Hopf bifurcation parameters that lead to machine tool chatter see Gilsinn [6].

In studying the effects of chatter it is sometimes desirable to compute the amplitude and frequency of the limit cycle generating the chatter. This entails solving the delay differential equations that model the machine tool dynamics. There is a large literature on numerically solving delay differential equations. Some representative methods are described in Banks and Kappel [10], Engelborghs and Luzyanina [11], Kemper [12], Paul [13], Shampine and Thompson [14], and Willé and Baker [15]. Although these methods generate solution vectors that can be studied by harmonic and power spectral methods to estimate the frequency of periodic cycles, they do not directly generate a representative model of a limit cycle, such as a Fourier series representation.

It is also desirable to know whether a representation of an approximate limit cycle is close to a true limit cycle. In other words we wish to answer the question as to whether the approximate solution represents sufficiently well a true solution. This is answered with a test criteria by Gilsinn [16], who showed that, given a representative approximate solution and frequency for a periodic solution to the autonomous delay differential equation

$$\dot{x} = X(x(t), x(t-h)), \tag{1}$$

where $x, X \in \mathbf{C}^n$, the space of n-dimensional complex numbers, $h > 0$, X sufficiently differentiable, there are conditions, depending on a non-criticality condition (to be defined below) and a number of parameters, for which (1) has a unique exact periodic solution and frequency in a numerically computable neighborhood of the approximate solution and frequency. This result was first established in a very general manner for functional differential equations by Stokes [17] who extended an earlier result for ordinary differential equations in Stokes [18]. However, no computable algorithms were given in the case of functional differential equations to estimate the various parameters. Only recently have algorithms been developed to computationally verify these conditions in the fixed delay case. A preliminary announcement of algorithms for computing these parameters was given by Gilsinn [19]. In this paper we include a more detailed discussion of the algorithms and apply them to a Van der Pol equation with delay in its non-linear terms.

The notation used in the paper is described in Section 2. The non-criticality condition is defined in Section 3. In Section 4 we construct an exact frequency and 2π-periodic solution of (1) as a perturbation problem. The main contraction theorem is proven in Section 5. In Sections 6–10 the necessary algorithms needed to compute the critical parameters for verifying the existence of a 2π-periodic solution of (1) will be given. In particular, a Galerkin based algorithm for approximating a periodic solution to (1) is given in Section 6. The general Floquet theory for DDEs is described in Section 7. An algorithm for computing the characteristic multipliers of the variational equation of (1) with respect to the approximate 2π-periodic solution, is given in Section 8. An algorithm to determine the solution to the formal adjoint equation with respect to the variational equation of (1) with respect to the approximate 2π-periodic solution, is given in Section 9. An algorithm for estimating a critical parameter, M, is given in Section 10. An application of these algorithms to the Van der Pol equation with delay is given in Section 11. The derivation of the coefficients for the pseudospectral differentiation matrix (73) is given in the Appendix.

2. Notation

Let C_ω denote the space of continuous functions from $[-\omega, 0]$ to $\mathbf{C}^n$ with norm in C_ω given by $|\phi| = \max |\phi(s)|$ for $-\omega \leqslant s \leqslant 0$, where

$$|\phi(s)| = \left(\sum_{i=1}^{n} |\phi_i(s)|^2 \right)^{1/2}. \tag{2}$$

C_ω is a Banach space with respect to this norm. We will sometimes use the notation $C_\omega(a)$ to denote the space of continuous functions on $[a-\omega, a]$. Let $\mathscr{P}$ be the space of continuous 2π-periodic functions with sup norm, $|\cdot|$ on $(-\infty, \infty)$. Let $\mathscr{P}_1 \subset \mathscr{P}$ be the subspace of continuously differentiable 2π-periodic functions with the sup