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The second law of thermodynamics and multifractal distribution functions: Bin counting, pair correlations, and the Kaplan–Yorke conjecture

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Abstract

We explore and compare numerical methods for the determination of multifractal dimensions for a doubly-thermostatted harmonic oscillator. The equations of motion are continuous and time-reversible. *At* equilibrium the distribution is a four-dimensional Gaussian, so that all the dimension calculations can be carried out analytically. *Away from* equilibrium the distribution is a surprisingly isotropic multifractal strange attractor, with the various fractal dimensionalities in the range $1 < D < 4$. The attractor is relatively homogeneous, with projected two-dimensional information and correlation dimensions which are nearly independent of direction. Our data indicate that the Kaplan–Yorke conjecture (for the information dimension) fails in the full four-dimensional phase space. We also find no plausible extension of this conjecture to the *projected* fractal dimensions of the oscillator. The projected growth rate associated with the largest Lyapunov exponent is *negative* in the one-dimensional coordinate space.
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1. Introduction

In 1983, Shuichi Nosé discovered a deterministic and time-reversible thermostatted dynamics capable of imposing a time-averaged kinetic temperature $\langle T \rangle$ on selected degrees of freedom [1,2]. His dynamics was both time-reversible and deterministic, but could nevertheless be used to model irreversible behavior. The most useful form of his dynamics has been called "Nosé–Hoover dynamics", after the studies of a thermostatted harmonic oscillator inspired by Nosé's work [3,4].

In 1987 related studies of the Galton Board [5] and Galton Staircase [6] models showed that *nonequilibrium* stationary states generated with time-reversible motion equations generate multifractal phase-space distributions [5–8]. A typical sample, from the Galton Board studies, is shown in Fig. 1. A dynamical analysis, through the Lyapunov spectrum, shows how symmetry breaking, through dynamical instability, results in trajectories obeying the Second Law of Thermodynamics. The motion *forward* in time is more stable (smaller Lyapunov exponents) than is the reversed motion *backward* in time. The resulting fractal distributions thus provide a simple resolution of the Loschmidt paradox, which contrasts the one-way Second Law of Thermodynamics with the either-way nature of time-reversible microscopic dynamics [6,9].

Here we investigate a prototypical nonequilibrium problem which generates a multifractal strange attractor in its (four-dimensional) phase space. The adjective "Multifractal" signifies that the apparent dimensionality of the attractor (the number of attractor points lying within a distance r is proportional to r^D) varies from point to point, making it possible to define whole families of fractal dimensions. Measures proportional to different powers of the phase-space probability density emphasize different regions of the attractor, giving rise to different characteristic overall dimensionalities. The dimensionalities of the nonequilibrium fractal distributions are

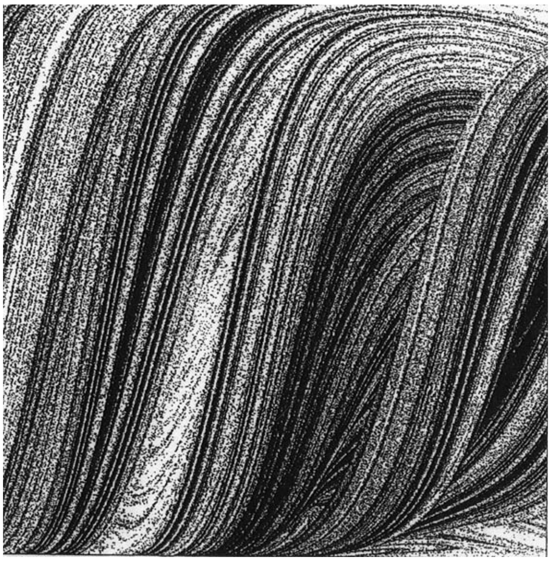

Fig. 1. Phase plane for the Galton Board problem. The 10^6 points shown here represent successive collisions of a point particle in an infinite periodic array of hard-disk scatterers. For details see Ref. [5].