Available online at www.sciencedirect.com
ScienceDirect
Communications in Nonlinear Science and Numerical Simulation 14 (2009) 379–398
Communications in Nonlinear Science and Numerical Simulation
www.elsevier.com/locate/cnsns

A high-order compact difference scheme for 2D Laplace and Poisson equations in non-uniform grid systems

Xuan Wang [a,*], Zhifeng Yang [a], Gordon Huang [b], Bin Chen [a]

[a] State Key Laboratory of Water Environment Simulation, School of Environment, Beijing Normal University, Beijing 100875, China
[b] Faculty of Engineering, University of Regina, Regina, Sask, Canada S4S 0A2

Received 19 February 2007; received in revised form 20 June 2007; accepted 11 July 2007
Available online 5 November 2007

Abstract

In this study, a high-order compact scheme for 2D Laplace and Poisson equations under a non-uniform grid setting is developed. Based on the optimal difference method, a nine-point compact difference scheme is generated. Difference coefficients at each grid point and source term are derived. This is accomplished through the consideration of compatibility between the partial differential equation and its difference discretization. Theoretically, the proposed scheme has third- to fourth-order accuracy; its fourth-order accuracy is achieved under uniform grid settings. Two examples are provided to examine performance of the proposed scheme. Compared with the traditional five-point difference scheme, the proposed scheme can produce more accurate results with faster convergence. Another reference scheme with the same nine-point grid stencil is derived based on the five-point scheme. The two nine-point schemes have the same coefficients for each grid points; however, their coefficients for the source term are different. The overall accuracy level of the solution resulting from the proposed scheme is higher than that of the nine-point reference scheme. It is also indicated that the smoothness of grids has significant effects on accuracy and convergence of the solutions; efforts in optimizing the grid configuration and allocation can improve solution accuracy and efficiency. Consequently, with the proposed method, solution under the non-uniform grid setting with appropriate grid allocation would be more accurate than that under the uniform-grid manipulation, with the same number of grid points.

PACS: 47.11.−j

Keywords: Laplace equation; Poisson equation; Difference scheme; Non-uniform grid system; High accuracy

1. Introduction

In studies of fluid flow and heat/mass transfer, Laplace and Poisson equations are generated to present problems like incompressible flow, heat equilibrium, and pollutant diffusion [1]. As one of the most effective

* Corresponding author. Tel.: +86 10 58800830; fax: +86 10 58800397.
E-mail address: wangx@bnu.edu.cn (X. Wang).

1007-5704/$ - see front matter
doi:10.1016/j.cnsns.2007.07.017

numerical approaches to solve these equations, the high-order compact (HOC) schemes have received much attention due to their advantages in solution accuracy and time requirement [2–7]. The scheme is "compact" in the sense that it involves only cells adjacent to a given node in a grid system; its stencil is smaller than those of the traditional difference schemes [2–7]. This result in nominal increase in arithmetic operations associated with an increased nodal accuracy in computing the stencil coefficients [2,3,7].

Several HOC schemes for the Laplace and Poisson equations were presented in [7–9]. For 2D and 3D problems, these schemes were developed through square- and cube-grid systems, respective, i.e. identical grid spacing is used for all axes. Another approach for solving the 2D Laplace equation was the optimal difference method (ODM) [10]. The ODM was valid for rectangular cells with an arbitrary length-to-width ratio, and might be viewed as an extension of the schemes as presented in [8,9]. However, similar to most of the existing HOC schemes for various differential equations [2–7], the formula produced by the ODM was limited to uniform grids with identical spacing in each axis (although the spacings in the two axes could be different from each other).

In the fields of hydraulic and environmental engineering, many large-scale or complex flow problems exist. Agrawal and Peckover [11] pointed out that, theoretically, difference schemes with uniform grids were usually simplest and most accurate. A uniform grid, however, was not suitable for complicated problems such as sharp variations of temperature (or concentration) over a short spatial distance. Under such circumstances, many more grid points would be needed, leading to prohibitively expensive and wasteful computational efforts [11]. An ideal way to improve the computational efficiency is to use a non-uniform scheme with high-resolution grids for high-variability areas and low-resolution ones for low-variability zones [12]. The schemes that employ non-uniform grids are also more suitable for natural boundary than those with uniform grids. In [13], the first-order formulas were used to discretize both the first- and second-order derivatives contained in one-dimensional steady convection–diffusion problems. It was concluded that, the use of a lower-order discretization formula on a non-uniform grid produced much better results than a higher-order formula on uniform grids [13]. It is of no doubt that, when the HOC schemes are applied to problems discretized with non-uniform grid distributions, the computational efficiency can be well maintained. A three-point non-uniform combined compact difference (NCCD) scheme with a global Hermitan polynomial spline was proposed for Stommel ocean models by Chu and Fan [12], with satisfactory results being obtained. The HOC schemes for 1D and 2D convection–diffusion problems were also extended to non-uniform grids [14]. It was shown that the HOC schemes could significantly improve solution accuracy for the convection–diffusion problem under a non-uniform grid setting [14]. For large-scale real-world problems, it is desired that schemes with non-uniform grids be employed to reduce computational cost and improve accuracy level.

Although the HOC schemes with uniform grids have been successfully used in fluid dynamics [2–10], they usually generated lower-than-expected accuracy and stability when being applied to non-uniform grids [11,15]. Another disadvantage of the HOC schemes for non-uniform grids is the associated algebraic complexity. Partly for this reason, it appears that these HOC schemes have seldom been extended to non-uniform grids, even though this is the avenue of greatest promise [14]. The ODM was introduced to develop the HOC scheme under a non-uniform grid setting for the 1D unsteady convection–diffusion problem and that for the 1D nonlinear convection–diffusion problem, respectively in [16] and [17]. As for a transformation was firstly implemented to eliminate the difficulty in dealing with the convection term; then the convection–diffusion equation was changed into a diffusion equation. Based on the HOC scheme for diffusion equation proposed by Wang and Yang [18], the corresponding HOC schemes for convection–diffusion equation could then be obtained through the converse transformation. The third- to the fourth-order formal accuracy and an unconditional stability were achieved for these parabolic and hyperbolic equations. For these schemes, non-uniform grids were produced through changes of grid steps. Compared to coordinate transformation which often fails to provide satisfactory results in the zone where sharp variations of physical scale exist, this kind of method is more satisfactory and easy to operate. Further extensions to second- or higher-dimensional cases are desired. Therefore, the objective of this study is to develop a HOC scheme for 2D Laplace and Poisson equations under a non-uniform grid setting.

In hydraulic and environmental engineering, the Laplace and Poisson equations are often used to describe equilibrium phenomena for many variables such as temperature and concentration. A two-dimensional Poisson equation can be written as follows: