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A modified tanh–coth method for solving the KdV and the KdV–Burgers' equations

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Abstract

In this work we use a modified tanh–coth method to solve the Korteweg-de Vries and Korteweg-de Vries–Burgers' equations. The main idea is to take full advantage of the Riccati equation that the tanh-function satisfies. New multiple travelling wave solutions are obtained for the Korteweg-de Vries and Korteweg-de Vries–Burgers' equations.

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1. Introduction

In this paper, we establish new travelling wave solutions to the Korteweg-de Vries (KdV) and Korteweg-de Vries–Burgers' (KdVB) equations given by

$$u_t + \alpha u_x + \beta u_{xxx} = 0 \tag{1}$$

and

$$u_t + uu_x - \alpha u_{xx} + \beta u_{xxx} = 0, \tag{2}$$

respectively, where α and β are some positive constants. The KdV equation (1) derived in 1895, see [1], models one-dimensional shallow water waves with small but finite amplitudes. It has also been used to describe a number of important physical phenomena such as magnetohydrodynamics waves in a warm plasma, acoustic waves in an anharmonic crystal and ion-acoustic waves [2]. Some papers, exploring various aspects of the above, can be found in [3–8].

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The Korteweg-de Vries–Burgers' equation (2) arises in many different physical contexts as a model equation incorporating the effects of dispersion, dissipation and nonlinearity [9]. Some examples are provided by the propagations of waves on an elastic tube filled with a viscous fluid [10], the flow of liquids containing gas bubbles [11] and turbulence [12].

Finding exact solutions of nonlinear partial differential equations (PDE's) has become more attractive subject due to the widespread of computer algebraic system (CAS), such as Maple and Mathematica. CAS allows us to do tedious and lengthy manipulations. Moreover, CAS can help us find new exact solutions of nonlinear PDE's.

Many methods were used to obtain travelling solitary wave solutions to nonlinear PDE's, such as the inverse scattering method [13–15], Hirota's bilinear method [16,17], the tanh method [18,19], the sine–cosine method [20,21], Backlund transformation method [22,23], the homogeneous balance [24,25], Darboux transformation [26], the Jacobi elliptic function expansion method [27].

Among those, the tanh method, established by Malfliet [18], uses a particularly straightforward and effective algorithm to obtain solutions for a large numbers of nonlinear PDE's. In recent years, much research work has been concentrated on the various extensions and applications of the tanh method. Fan [28,29] has proposed an extended tanh method and obtained new travelling wave solutions that cannot be obtained by the tanh method. Recently, Wazwaz extended the tanh method and call it first the extended tanh method [30–32] and later as the tanh–coth method [33]. Most recently, El-Wakil [34,35] and Soliman [36] modified the extended tanh method (the tanh–coth method) and obtained new solutions for some nonlinear PDE's. The goal of this work is to implement the tanh–coth method and the Riccati equation in [37] to obtain more new exact travelling wave solutions of the KdV and KdV–Burgers' equations.

2. Description of the method

Consider the general nonlinear wave PDE's, say, in two variables:

$$u_t = G(u, u_x, u_{xx}, \ldots). \tag{3}$$

In order to apply the tanh–coth method, the independent variables, x and t, are combined into a new variable, $\xi = \kappa(x - \omega t)$, where κ and ω represent the wave number and velocity of the travelling wave, respectively. Both are undetermined parameters with the assumption that $\kappa > 0$. Therefore, $u(x,t)$ is replaced by $u(\xi)$, which defines the travelling wave solutions of Eq. (3). Equations such as Eq. (3) are then transformed into

$$-\kappa\omega \frac{du}{d\xi} = G\left(u, \kappa\frac{du}{d\xi}, \kappa^2\frac{d^2u}{d\xi^2}, \ldots\right). \tag{4}$$

Hence, under the transformation $\xi = \kappa(x - \omega t)$, the PDE in Eq. (3) has been reduced to an ordinary differential equation (ODE) given by Eq. (4). The resulting ODE is then solved by the tanh–coth method [27], which admits the use of a finite series of functions of the form:

$$u(x,t) = u(\xi) = a_0 + \sum_{j=1}^{n}[a_j Y^j(\xi) + b_j Y^{-j}(\xi)] \tag{5}$$

and the Riccati equation

$$Y' = A + BY + CY^2, \tag{6}$$

where $\prime := \frac{d}{d\xi}$, and A, B, and C are constants to be prescribed later. The parameter n is a positive constant that can be determined by balancing the linear term of highest order with the nonlinear term in Eq. (4). Inserting Eq. (5) into the ODE in Eq. (4) and using Eq. (6) results in an algebraic equation in powers of Y. Since all coefficients of Y^j must vanish. This will give a system of algebraic equations with respect to parameters a_i, b_i, κ and ω. With the aid of Maple, we can determine a_i, b_i, κ and ω. We will consider the following special solutions of the Riccati equation (6):

(I) $A = B = 1$ and $C = 0$, Eq. (6) has the solution $Y = e^{\xi} - 1$.
(II) $A = 1/2$, $B = 0$ and $C = -1/2$, Eq. (6) has the solutions $Y = \coth\xi \pm \operatorname{csch}\xi$ and $Y = \tanh\xi \pm i\operatorname{sech}\xi$, where $i^2 = -1$.

To illustrate the method, we consider the KdV and the KdV–Burgers' equations below.