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Analytical reconstruction of isotropic turbulence spectra based on the Gaussian transform

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ABSTRACT

The field of application of the Random Particle Mesh (RPM) method used to simulate turbulence-induced broadband noise in several aeroacoustic applications is improved to realise isotropic turbulence spectra. With this method turbulent fluctuations are synthesised by filtering white noise with a *Gaussian* filter kernel that in turn gives a *Gaussian* spectrum. The *Gaussian* filter is efficient and finds wide-spread applications in stochastic signal processing. However *Gaussian* spectra do not correspond to real turbulence spectra. Thus in turbo-machines the *von Kármán*, *Liepmann*, and *modified von Kármán* spectra are more realistic model spectra. In this note we analytically derive weighting functions to realise arbitrary isotropic solenoidal spectra using a superposition of weighted *Gaussian* spectra of different length scales. The analytic weighting functions for the *von Kármán*, the *Liepmann*, and the *modified von Kármán* spectra are derived subsequently. Finally a method is proposed to discretise the problem using a limited number of *Gaussian* spectra. The effectivity of this approach is demonstrated by realising a *von Kármán* velocity spectrum using the RPM method.

1. Introduction

A stochastic noise signal of a certain spectral shape can be generated by convolution of a white noise signal by a filter kernel of an appropriate shape [1].

One of the most common filter kernels is the *Gaussian* filter kernel that realises a *Gaussian* spectrum. The *Gaussian* filter is very simple and time efficient as it has benefitial characteristics: its derivatives and integrals are again *Gaussian* functions; the filter is separable along each spatial dimension so that it can be applied to each dimension independently; fast filter methods are available, such as Purser [2] and Young and Van-Vliet [3] filters.

But *Gaussian* spectra seldom represent the physics of turbulence. Here more elaborate spectra are needed, such as *Kolmogorov*, *von Kármán* or *Liepmann* spectra. For these spectra the filter kernels are very complicated to use and they are fully coupled in space, as shown by Dieste and Gabard [4].

Siefert et al. [5] succeeded in realising the *Kolmogorov* spectrum using the superposition of weighted *Gaussian* spectra. The weight of each *Gaussian* spectrum was optimised empirically to fit the target spectrum. Others have adopted this method for other kinds of spectra, e.g., just recently Gea-Aguilera et al. [6] and Kim et al. [7] published their findings. Note that the method introduced here has already been presented by the authors on conferences but not derived in detail [8,9].

The objective of this note is to provide a theoretical background for determining the appropriate analytical weighting function by means of *Gaussian* transform [10]. The analytical weighting function is derived for the *von Kármán*, the *Liepmann* and the *modified von Kármán* spectra. Furthermore, an efficient method is proposed to discretise the weighting function with a limited number of *Gaussian* spectra. Suggestions are made to choose the number of filters and their length scales. As illustration, the realised velocity spectrum using the RPM method [1] is compared to the analytically derived velocity spectrum.

2. Method - Gaussian transformation

The weighting functions are derived in the three dimensional (3D) space. As it will be briefly discussed in Section 2.4 the derived weighting functions can be applied to the two dimensional (2D) space without modification. The realised energy and velocity spectra differ in 2D and 3D in agreement to the theory [11].

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2.1. Turbulence spectra

The most popular models for isotropic turbulence are the *von Kármán*, *Liepmann*, and *modified von Kármán* models [12].

2.1.1. The von Kármán spectrum

The *von Kármán* spectrum is commonly used to represent homogeneous isotropic turbulence. It satisfies the energy law distribution of k^4 for the large eddies which contain most of the energy and reproduces the $k^{-5/3}$ gradient in the inertial subrange. The energy spectrum is given by

$$E_K(\hat{k}) = \frac{55}{9\pi} u_t^2 \Lambda \frac{\hat{k}^4}{(1+\hat{k}^2)^{17/6}}, \tag{1}$$

with the mean turbulent velocity u_t related to the turbulent intensity T_u and the mean flow velocity u_0 by $u_t^2 = (T_u \cdot u_0)^2$, the integral length scale Λ, and the reduced wavenumber $\hat{k}$ defined by $\hat{k} = k^*/k_e$, where $k^* = k\Lambda$, and $k_e = \frac{\sqrt{\pi}\Gamma(5/6)}{\Gamma(1/3)}$.

2.1.2. The Liepmann spectrum

Liepmann determined turbulence longitudinal correlation coefficients from measurements and found that they can be approximated by an exponential law $f(x) = \exp\left(\frac{-x}{\Lambda}\right)$. The resulting model spectrum is given as [12,13]:

$$E_L(k^*) = \frac{8u_t^2\Lambda}{\pi}\frac{{k^*}^4}{(1+{k^*}^2)^3}. \tag{2}$$

The *Liepmann* spectrum is comparable in shape to the *von Kármán* spectrum.

2.1.3. The modified von Kármán spectrum

According to Bechara [14] the *von Kármán* spectrum can be *modified* to be representative over the entire wavenumber range including the dissipation subrange

$$E_M(\hat{k}) = E_K(\hat{k}) \exp\left(-2\frac{k^2}{k_d^2}\right) \tag{3}$$

with the *Kolmogorov* wavenumber $k_d = \left(\frac{\epsilon}{\nu^3}\right)^{1/4}$, where ϵ is the specific dissipation rate and ν is the eddy viscosity.

2.2. Weighting function

According to Ewert et al. [1] filtering of a white noise field with a *Gaussian* filter kernel of a specific length scale realises a *Gaussian* spectrum of the form

$$E_G(k) = \frac{4u_t^2\Lambda}{\pi^3}{k^*}^4 e^{\frac{-{k^*}^2}{\pi}}. \tag{4}$$

For convenience we introduce a new spectrum e_G such that its integral over the wavenumber range is one, i.e.

$$\int_0^\infty e_G(k)\mathrm{d}k = 1 \qquad \Rightarrow E_G(k) = \frac{3}{2}u_t^2 e_G(k). \tag{5}$$

We are looking for a weighting function $f(l, \Lambda)$ to realise an arbitrary spectrum $e(k)$ of integral length scale Λ by means of a superposition of *Gaussian* spectra e_G of length scales l:

$$e(k, \Lambda) = \int_0^\infty f(l, \Lambda) e_G(k, l)\mathrm{d}l.$$

Using Eqs. (4) and (5) yields the following solution:

$$e(k, \Lambda) = \int_0^\infty f(l, \Lambda)\frac{8}{3\pi^3} l^5 k^4 \exp\left(-\frac{k^2 l^2}{\pi}\right)\mathrm{d}l. \tag{6}$$

Note that only the weighting function $f(l, \Lambda)$ depends on the integral length scale Λ. In the following we drop Λ in the expression of the weighting function f and write $f(l, \Lambda) = f(l)$.

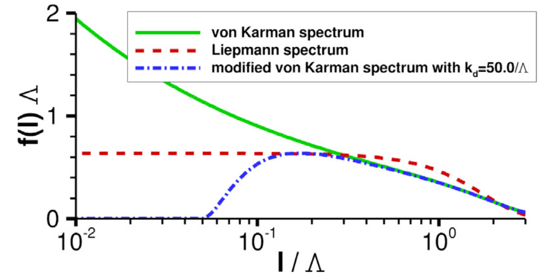

Fig. 1. The weighting function $f(l)$ is used to realise typical turbulence spectra of length scale Λ using a superposition of *Gaussian* spectra of various length scales l. The analytical weighting functions for the *von Kármán* (solid), the *Liepmann* (dashed), and the *modified von Kármán* (dash-dot) spectra are shown.

A parameter σ is introduced to write Eq. (6) in a suitable manner for *Gaussian* transform as defined by Alecu et al. [10]. This parameter verifies the two following relationships:

$$l^2 = \frac{\pi}{2\sigma^2} \qquad \text{and} \qquad \frac{\mathrm{d}l}{\mathrm{d}\sigma^2} = -\frac{\sqrt{\pi}}{2\sqrt{2}\sigma^3}.$$

Eq. (6) rewrites:

$$\underbrace{\frac{e(k)}{k^4}}_{p(k)} = \int_0^\infty \underbrace{f\left(\sqrt{\frac{\pi}{2}\frac{1}{\sigma}}\right)\frac{\sqrt{\pi}}{3\sqrt{2}\sigma^7}}_{G(\sigma^2)} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{k^2}{2\sigma^2}\right)}_{\mathbb{N}(k|\sigma^2)}\mathrm{d}\sigma^2. \tag{7}$$

According to Alecu et al., $p(k)$ is a zero-mean generic symmetric distribution, $G(\sigma^2)$ is the mixture function and $\mathbb{N}(k|\sigma^2)$ is the zero-mean *Gaussian* distribution. The *Gaussian* transform $\mathcal{G}$ is defined as the operator which transforms $p(k)$ into $G(\sigma^2)$. The inverse *Gaussian* transform $\mathcal{G}^{-1}(G(\sigma^2)) = p(k)$ is simply given by Eq. (7).

From Eq. (7) the weighting function $f(l)$ is given as

$$f\left(l = \sqrt{\frac{\pi}{2}\frac{1}{\sigma}}\right) = \frac{3\sqrt{2}\sigma^7}{\sqrt{\pi}}G(\sigma^2). \tag{8}$$

2.2.1. Von Kármán weighting function

With the *von Kármán* spectrum given in Eq. (1) the left-hand side of Eq. (7) becomes

$$p(k) = \frac{110}{27\pi}\Lambda^5 k_e^{5/3}\frac{1}{(k_e^2 + k^2\Lambda^2)^{17/6}}. \tag{9}$$

The direct *Gaussian* transform is given by Alecu et al. [10, Eq.(4)]:

$$G(\sigma^2) = \mathcal{G}(p(k)) = \frac{1}{\sigma^2}\sqrt{\frac{\pi}{2\sigma^2}}\left(\mathcal{L}^{-1}\left(p(\sqrt{s})\right)(t)\right)_{t=\frac{1}{2\sigma^2}}, \tag{10}$$

where $\mathcal{L}^{-1}$ is the inverse *Laplace* transform. Using the relation

$$\mathcal{L}^{-1}\left(\frac{1}{(p-\alpha)^n}\right)(t) = \frac{e^{\alpha t}t^{n-1}}{\Gamma(n)}, \tag{11}$$

where $\Gamma(n) = (n-1)!$ is the gamma function, we find for Eq. (9)

$$\mathcal{G}(p(k)) = \frac{55}{54\sqrt{\pi}\Gamma(17/6)}\sqrt[3]{\frac{k_e^5}{2\sigma^{20}\Lambda^2}}\exp\left(-\frac{k_e^2}{2\Lambda^2\sigma^2}\right) \tag{12}$$

and the weighting function for the *von Kármán* spectrum is given by

$$\boxed{f_K(l) = \frac{55}{18\Gamma(17/6)\sqrt{\pi}}\sqrt[3]{\frac{k_e^5}{\pi\Lambda^2 l}}\exp\left(-\frac{k_e^2 l^2}{\pi\Lambda^2}\right).} \tag{13}$$

Fig. 1 shows the corresponding weighting function.