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Finite-volume scheme for the solution of integral boundary layer equations

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ABSTRACT

An unstructured-mesh finite-volume formulation for the solution of systems of steady conservation laws on embedded surfaces is presented. The formulation is invariant to the choice of local tangential coordinate systems and is stabilized by a novel up-winding scheme applicable also to mixed-hyperbolic systems. The formulation results in a system of non-linear equations which is solved by a quasi-Newton method. While the finite volume scheme is applicable to a range of conservation laws, it is here implemented for the solution of the integral boundary layer equations, as a first step in developing a fully coupled viscous-inviscid interaction method. For validation purposes, integral boundary layer quantities computed using a minimal set of three-dimensional turbulent integral boundary layer equations are compared to experimental data and an established computer code for two-dimensional problems. The validation shows that the proposed formulation is stable, yields a well-conditioned global Jacobian, is conservative on curved surfaces and invariant to rotation as well as convergent with regard to mesh refinement.

1. Introduction

Viscous-inviscid interaction (VII) methods allow computation of viscous flow properties with relatively high accuracy at a relatively low computational cost. For aeronautical applications, integral VII methods are currently used for prediction of viscous flow properties over two dimensional airfoil sections and are able to maintain adequate solution accuracy even in the somewhat challenging regime of transitional flows involving laminar separation bubbles [1]. Some three-dimensional inviscid flow solvers have been extended with approximate boundary layer analysis methods which solve for two-dimensional boundary layer development along surface streamlines or slices of a structured mesh [2,3]. Due to the simplifications involved, such an approximate solution cannot account for three-dimensional effects such as cross-flow within the boundary layer. Some applications, such as the aerodynamic design of wing–fuselage fairings, benefit substantially from fully accounting for the three-dimensional boundary layer development [4]. A number of attempts have been made to develop a fully three-dimensional VII scheme, a review is given by van Garell [5], but, to the authors' knowledge, for general, large scale, three-dimensional problems defined on complex geometries with unstructured meshes such a scheme is not readily available at present.

As a step towards developing a robust fully three-dimensional VII method applicable to whole aircraft configurations the current study describes the development of a surface discretization for solution of steady conservation laws. There are a few things which need to be considered. The integral boundary layer (IBL) equations have a complex character which can change between hyperbolic and mixed-hyperbolic depending on the chosen closure relations [6]. The integral equations are obtained by wall-normal integration of the corresponding differential equations and for curved surfaces the integral equations are thus defined on a two-parametric surface in three space dimensions. Because of the reduced mesh generation effort, it is desirable that the discretization scheme is applicable to unstructured meshes. Together, these aspects set high standards for the discretization.

Based on a local Cartesian formulation [7], Mughal, Nishida and Drela have applied modified Galerkin finite-element methods to discretize different sets of IBL equations [6,8,9]. Some of these formulations bear resemblance to a streamline-upwind Petrov-Galerkin approach [10]. This method is well established, but requires that an upwind direction can be defined unambiguously and also the tuning of a scalar stabilization parameter. It was found rather difficult to robustly define both direction and stabilization parameter for general three-dimensional problems on arbitrary surfaces. Later work by Drela [9] makes use of a standard Galerkin finite-element method which is stabilized by the addition

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Nomenclature

Coordinate systems

G	Global Cartesian coordinate system
C	Local Cartesian coordinate system
$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$	Cartesian coordinate vectors
$\boldsymbol{s}, \boldsymbol{c}, \boldsymbol{n}$	Streamline based Cartesian coordinate vectors

Symbols related to discretization

$\boldsymbol{u}$	Nodal variables
$\boldsymbol{F}$	Flux function
$\boldsymbol{g}$	Source function
$\boldsymbol{f}$	Flow vector
$\boldsymbol{n}$	Normal vector
$\boldsymbol{S}$	Cell face area
V	Cell volume
$\boldsymbol{M}$	Rotation matrix
$\boldsymbol{T}$	Rotation matrix (also row permutation matrix)
$\boldsymbol{t}$	Rotation axis
α_{ij}	Angle between cell normals i, j
$\boldsymbol{C}$	Barycenter of mesh triangle
$\boldsymbol{P}$	Edge mid-point of mesh triangle
$\boldsymbol{r}$	Local residual vector
$\boldsymbol{R}$	Global residual vector (also right eigenvectors)

Symbols related to upwind-scheme

$\boldsymbol{\Lambda}$	Block-diagonal matrix with eigenvalues
$\boldsymbol{L}$	Left eigenvectors
$\boldsymbol{R}$	Right eigenvectors (also global residual vector)
$\boldsymbol{T}$	Row permutation matrix (also rotation matrix)
$\boldsymbol{w}$	Transformed nodal variables
$\boldsymbol{D}$	Diagonal matrix with weight factors
$\boldsymbol{A}$	Matrix
$\boldsymbol{B}$	Matrix
α	Weight factor
C	Condition number

Symbols related to integral boundary layer equations

θ	Momentum loss thickness
δ^*	Displacement thickness
θ^*	Energy deficit thickness
δ^{**}	Density thickness
τ	Shear stress
D	Dissipation
β_w	Cross flow angle
H_k	Kinematic shape factor
u, w	x-, z-components of velocity
q	Magnitude of velocity
ρ	Density
ν	Dynamic viscosity
Re	Reynolds number
X	Nodal quantity
e_X	Relative error

Subscripts

e	Value at edge of boundary layer
∞	Freestream value
1	Streamwise value
i, j	Cell index

Superscripts

(i), (j)	Quantity expressed in local Cartesian coordinate system i, j
(G)	Quantity expressed in global Cartesian coordinate system

of a symmetrically diffusive artificial dissipation term. The problem of selecting an appropriate stabilization (artificial dissipation) parameter can be circumvented by means of a least-squares finite element (LSFEM) approach [11]. Regrettably, earlier work by one of the present authors showed that, at least for standard formulations of the IBL equations, the LSFEM yields rather ill-conditioned problems due to the squaring of the residual terms. Furthermore, its was found difficult to treat domains with multiple inflow boundary conditions such as full aircraft configurations, as the dissipative characteristics of the LSFEM introduced too large errors. Closer examination shows that the desired solution for such a case is distinguished by large gradients of the boundary layer variables in the cross-flow direction, as the thick boundary layer developing along the fuselage meets a rather thin layer starting at the wing leading edge. The solution in these areas is thus severely affected by the dissipative characteristics.

Hyperbolic problems permitting discontinuities are often solved by means of a finite-volume scheme and such a scheme forms the basis for the present discretization. When applied to hyperbolic problems finite volume schemes can be made stable and well-conditioned by means of upwinding [12]. However, since the integral boundary layer equations can switch between a hyperbolic and a mixed-hyperbolic character [6] there is a need to employ a discretization which is stable also in mixed-hyperbolic regions. An important contribution of the present work is that it employs a novel approach to determine the required upwind-biased flux term for general forms of conservation and closure equations. The current discretization is based on a local Cartesian formulation [6]. For application on embedded surfaces there is thus a need to transform the flow leaving one control volume into an equivalent flow expressed in the local coordinate system of the neighboring control volume in a way which ensures flow conservation. In the current formulation this is achieved by a set of rotation matrices which will be described in detail.

It is to be noted that the discretization scheme in the present study is not limited to applications related to the integral boundary layer equations but are applicable to any system of steady conservation laws defined on an embedded surface.

2. Method

This section describes the formulation of the finite-volume scheme as well as its application for the solution of a set of three-dimensional IBL equations.

2.1. Finite volume discretization

Systems of conservation laws are typically written as

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla_s \cdot \boldsymbol{F}(\boldsymbol{u}) + \boldsymbol{g}(\boldsymbol{u}) = 0. \tag{1}$$

where, $\nabla_s \cdot \boldsymbol{F}$ represents the surface divergence of a flux function $\boldsymbol{F}$, while $\boldsymbol{g}$ describes a source function and $\boldsymbol{u}$ denotes the variables. For steady problems, which are the focus of the present study, the time derivative vanishes.

A finite-volume discretization of (1) is obtained by integration over a small control volume and application of Gauss' divergence theorem

$$\int_V \nabla_s \cdot \boldsymbol{F}(\boldsymbol{u}) + \boldsymbol{g}(\boldsymbol{u})\, dV = \int_S \boldsymbol{F}(\boldsymbol{u}) \cdot \boldsymbol{n}_s\, dS + \int_V \boldsymbol{g}(\boldsymbol{u})\, dV, \tag{2}$$

where V is a control volume and S its boundary. The first term on the right-hand side contains the flow $d\boldsymbol{f} = \boldsymbol{F} \cdot \boldsymbol{n}_s\, dS$ obtained by the product of the flux and the cell boundary normal $\boldsymbol{n}_s$, where the