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New nonlinear weights for improving accuracy and resolution of weighted compact nonlinear scheme

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ABSTRACT

This paper proposes a new kind of nonlinear weights to improve accuracy and resolution of high-order weighted compact nonlinear scheme. The new nonlinear weights are constructed based on not only the ratios between different smoothness indicators but also their values. The values of smoothness indicators are explicitly considered in the basic formulas of the new nonlinear weights after a careful analysis of convergence accuracy and shock capturing property. The new nonlinear weights approach to the optimal weights as the smoothness indicators approaching zero. Thus, the new nonlinear weights are close to the optimal weights in smooth regions where the smoothness indicators are small. Therefore, optimal order accuracy is maintained in smooth regions. In addition, near discontinuities the new nonlinear weights degenerate to the original nonlinear weights used for designing. Thus, discontinuity capturing ability is ensured. Numerical results show that the weighted compact nonlinear scheme with the new nonlinear weights achieves optimal order accuracy even near high-order critical points, captures discontinuities sharply without obvious oscillation, has higher resolution and higher efficiency than other nonlinear schemes and has obvious advantage in capturing small scale structures.

1. Introduction

High-order accurate and high-resolution schemes with discontinuity capturing ability are desired in simulating multi-scale flows which contain shock waves, such as direct numerical simulation (DNS) and large eddy simulation (LES) of high speed turbulence [1–4]. Many high-order discontinuity capturing schemes have been constructed. In 1980s, third-order essentially non-oscillatory (ENO) scheme was constructed by Harten and Engquist [5]. Later, Jiang and Shu [6] put forward weighted ENO (WENO) scheme by combining the weighting technique with the ENO scheme. Compared with the ENO scheme, the WENO scheme has similar discontinuity capturing property but higher-order accuracy and higher resolution. However, the WENO scheme proposed by Jiang and Shu [6] (WENO-JS) can not achieve optimal order accuracy near critical points of smooth solutions where some leading derivatives of the solution vanish [7,8].

To solve this problem, Henrick et al. [7] put forward a Mapped WENO scheme (WENO-M), which ensures optimal order accuracy at first-order critical points. The WENO-M scheme exhibits better resolution than the WENO-JS scheme, but its computation cost is about 25% higher than the latter. Borges and co-workers [8] solved this problem by proposing Z nonlinear weights. And the WENO scheme with the Z nonlinear weights (WENO-Z) can also achieve optimal order accuracy at first-order critical points [8,9]. Compared with the WENO-JS scheme, the WENO-Z scheme has similar computation cost, but can achieve higher resolution and capture discontinuities more sharply [8,9]. However, the WENO-M and WENO-Z schemes still suffer a loss in accuracy near second-order and higher-order critical points. To achieve optimal order accuracy near high-order critical points, Yamaleev and Carpenter [10,11] proposed some limitations of ϵ, a parameter originally introduced to avoid the denominator becoming zero. Castro, Don and coworkers [12,13] made some further studies on the limitations of ϵ for the WENO-Z scheme and proved that the WENO-Z scheme can achieve optimal order accuracy near high-order critical points if ϵ satisfies some carefully designed limitations.

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Many investigations on the ϵ have shown that it is far more than a parameter to avoid the denominator becoming zero, it also affects the accuracy and resolution of the nonlinear schemes. Arandiga et al. [14,15] and Kolb [16] have made some valuable work on ϵ based on the work of Yamaleev and Carpenter [10,11]. Osher and Fedkiw [17] have pointed out that ϵ is a dimensional quantity and have proposed a new formula for ϵ which scales consistently with the local flow variables. Henrick et al. have found that ϵ has a dramatic effect on the convergence order of the WENO-JS scheme near critical points [7]. Their results indicate that the nonlinear weights have completely different performances for the case that β_k is much smaller than or comparable to ϵ and for the case that β_k is much larger than ϵ. In another word, the values of β_k have been implicitly introduced into the nonlinear weights if ϵ is much larger than the machine zero. Thus, the nonlinear weights in [6,10–13] are actually dependent on both the ratios between and the values of β_k. There is also some other work that makes the nonlinear effects related to the values of β_k or some other undivided flow variable derivatives to improve some properties of nonlinear schemes, such as the studies in [18–20]. The numerical results in these papers have revealed the fact that considering the values of β_k properly can improve the properties of nonlinear schemes.

Traditionally, the nonlinear weights are designed to use the ratios between β_k to detect discontinuities. However, the ratios between β_k may also be very large in smooth regions, for example, near critical points. Thus these points may be treated like discontinuities, which may lead to a loss in accuracy. Note that the values of β_k in smooth regions are much smaller than those near discontinuities. Thus, it is possible for nonlinear weights to distinguish critical points from discontinuities by considering the values of β_k.

However, the properties (resolution, for example) of the nonlinear weights in [6,10–13] can be improved only in a limited part of the smooth regions because the values of β_k are effective only if β_k is much smaller than or comparable to ϵ and ϵ has to be small enough to restrict the oscillations which may appear near discontinuities. Based on these observations, making full use of β_k to improve the properties of nonlinear schemes is the main motivation of the current work.

In this paper, a completely new method is put forward to explicitly consider the values of β_k in the basic formulas of the nonlinear weights, which is different from previous work that introduces the values of β_k implicitly by ϵ [6,10–13]. Besides achieving optimal order accuracy, the resolution of the corresponding nonlinear scheme is improved in the major smooth regions rather than only in a limited part of the smooth regions. The present work is based on weighted compact nonlinear scheme (WCNS) [21] which also uses nonlinear weights similar to those of the WENO schemes. The WCNS can easily satisfy geometric conservation law (GCL) [22,23] and has superiority in simulations on complex grids [24]. The new nonlinear weights can also be extended to the WENO schemes. Some canonical cases, such as Osher–Shu problem double Mach problem, shock turbulence interaction problem are used to test the shock capturing ability, high frequency wave simulating ability and turbulence simulating ability of the WCNS with the new nonlinear weights. It is shown that the new scheme not only achieves optimal order accuracy and captures discontinuities without obvious oscillation, but also has higher resolution and obvious advantage in simulating turbulence.

The organization of the paper is as follows. The WCNS is introduced in Section 2 which also contains a convergence analysis and three kinds of nonlinear weights. In Section 3, the new nonlinear weights are put forward, and their properties are analyzed. Section 4 presents numerical experiments to verify the theoretical analysis. At last, conclusions are drawn in Section 5.

2. WCNS and it convergence analysis

In Section 2.1, the framework of the WCNS is introduced briefly. Then convergence order of the WCNS is analyzed and sufficient conditions for optimal order accuracy are derived in Section 2.2. Finally, three typical kinds of nonlinear weights are presented and analyzed in Section 2.3.

2.1. WCNS

Hyperbolic conservation law has the following form

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0, \tag{1}$$

where u is a conserved quantity, f describes its flux. Here, we restrict our discussion to the one dimensional scalar case.

Consider a uniform grid defined by $x_j = j\triangle x = jh$, $j = 0, \ldots, N$, where $\triangle x = h$ is the uniform grid spacing. The semi-discrete form of Eq. (1) yields an ordinary differential equation:

$$\frac{du_j(t)}{dt}=-F_j^{'}, \tag{2}$$

where $u_j(t)$ is a numerical approximation of $u(x_j, t)$, and $F_j^{'}$ is a spacial discretization of $\frac{\partial f}{\partial x}|_{x=x_j}= f_j^{'}$. In this paper, the WCNS is used for spatial discretization.

The WCNS [21] was proposed by combining the weighted nonlinear interpolation with central compact schemes of Lele [25]. The WCNS can satisfy GCL easily by using symmetrical conservative metric method (SCMM) [23] to calculate grid metrics, and numerical tests have shown that WCNS is robust and can give accurate results on complex grids [24,26]. The WCNS consists of three parts [21,27]: high-order flux difference scheme for flux derivatives, numerical flux construction for cell-edge fluxes and high-order interpolation for cell-edge variable values.

In the first part, hybrid cell-edge and cell-node compact scheme (HCS) in [27,28] is adopted to calculate flux derivatives. HCS is an extension of the cell-node mesh compact scheme and cell-centered mesh compact scheme of Lele [25]. It uses both cell-edge and cell-node values on the right hand side of the scheme, which results in better spectral properties [27,29,30]. The general form of HCS reads

$$\begin{aligned}&\gamma F_{j-2}^{'}+\chi F_{j-1}^{'}+F_j^{'}+\chi F_{j+1}^{'}+\gamma F_{j+2}^{'}\\&\quad=\frac{\varphi}{h}(\tilde F_{j+\frac12}-\tilde F_{j-\frac12})+\frac1h\sum_{m=1}^{3}a_m(f_{j+m}-f_{j-m}),\end{aligned} \tag{3}$$

where f_{j+m} is the flux at cell-node $j+m$,

$$\tilde F_{j+\frac12}=\tilde F(\tilde u^R_{j+\frac12},\tilde u^L_{j+\frac12}), \tag{4}$$

is the numerical flux at cell-edge $j+\frac12$, $\tilde u^R_{j+\frac12}$ and $\tilde u^L_{j+\frac12}$ are cell-edge variable values calculated by upwind-like nonlinear interpolations. The parameters in Eq. (3) determine the accuracy and spectral property of the HCS. In this paper, we consider a special case (HCS-E6)

$$\begin{aligned}F_j^{'}=&\frac{\varphi}{h}\left(\tilde F_{j+\frac12}-\tilde F_{j-\frac12}\right)+\frac{192-175\varphi}{256h}\left(f_{j+1}-f_{j-1}\right)\\&+\frac{-48+35\varphi}{320h}\left(f_{j+2}-f_{j-2}\right)+\frac{64-45\varphi}{3840h}\left(f_{j+3}-f_{j-3}\right).\end{aligned} \tag{5}$$

Replacing $\tilde F_{j\pm\frac12}$ by the exact fluxes, we can get the following estimation using Taylor series expansion

$$F_j^{'}=f_j^{'}+\left(\frac{1}{8960}-\frac{5}{65536}\varphi\right)f_j^{(6)}h^6+O(h^8). \tag{6}$$