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Computation of steady incompressible flows in unbounded domains

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ABSTRACT

In this study we revisit the problem of computing steady Navier–Stokes flows in two-dimensional unbounded domains. Precise quantitative characterization of such flows in the high-Reynolds number limit remains an open problem of theoretical fluid dynamics. Following a review of key mathematical properties of such solutions related to the slow decay of the velocity field at large distances from the obstacle, we develop and carefully validate a spectrally-accurate computational approach which ensures the correct behavior of the solution at infinity. In the proposed method the numerical solution is defined on the entire unbounded domain without the need to truncate this domain to a finite box with some artificial boundary conditions prescribed at its boundaries. Since our approach relies on the streamfunction–vorticity formulation, the main complication is the presence of a discontinuity in the streamfunction field at infinity which is related to the slow decay of this field. We demonstrate how this difficulty can be overcome by reformulating the problem using a suitable background "skeleton" field expressed in terms of the corresponding Oseen flow combined with spectral filtering. The method is thoroughly validated for Reynolds numbers spanning two orders of magnitude with the results comparing favorably against known theoretical predictions and the data available in the literature.

1. Introduction

In this work we revisit the classical problem of computing steady flows past an obstacle in an unbounded domain which has played an important role in theoretical fluid mechanics, especially, in the study of separated flows [49]. An aspect of this problem which has received particular attention is the structure of the flow field in the limit when the Reynolds number $Re \to \infty$. It is well known that the inviscid Euler flows in the same geometric setting admit several different solutions with quite distinct properties – in addition to the Kirchhoff free-streamline flows featuring an open wake region extending to infinity [43,12], flows with compact vorticity regions predicted by the Prandtl–Batchelor theory [4] have also been found [20]. Perturbation-type solutions to this problem were constructed using methods of asymptotic analysis by Chernyshenko [14,15]. While these solutions remain the most advanced theoretical results concerning this problem, their computational validation for large Re remains an open problem with Fornberg's results from the late 1980s still representing the state-of-the-art [24,25]. As will be argued below, what makes this problem challenging from the computational point of view is the combination of steadiness and an unbounded domain which results in a very slow decay of the flow fields towards their limiting values at large distances from the obstacle. In the recent years significant advances have been made as regards mathematical characterization of such flows [30], and the goal of this work is to develop and validate a numerical approach which explicitly accounts for these properties. More specifically, the proposed technique will achieve the spectral accuracy for solutions defined on unbounded domains (i.e., without the need to truncate the domain to a finite "computational box" with some artificial boundary conditions prescribed on its boundaries) and will in addition ensure that solutions have the right asymptotic behavior at large distances from the obstacle.

We thus consider the problem defined on the two-dimensional (2D) unbounded domain Ω which is the exterior of a circular obstacle A of diameter d (Fig. 1). Given the free stream velocity at infinity U_∞, the system of equations we are interested in is

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \frac{1}{Re}\Delta\mathbf{v} \qquad \text{in } \Omega, \tag{1a}$$

$$\nabla\cdot\mathbf{v} = 0 \quad \text{in } \Omega, \tag{1b}$$

$$\mathbf{v} = \mathbf{0} \quad \text{on } \partial A, \tag{1c}$$

$$\mathbf{v} \to U_\infty \mathbf{e}_x \quad \text{as } |\mathbf{x}| \to \infty, \tag{1d}$$

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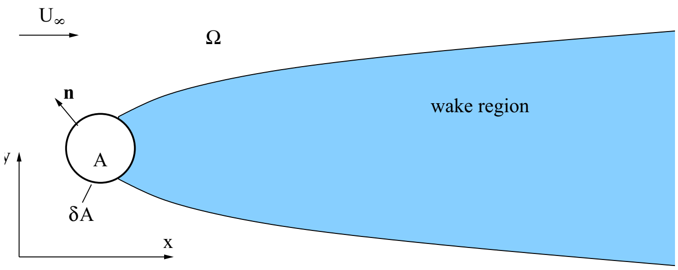

Fig. 1. Geometry of the flow domain Ω with a schematic representation of the wake region (shaded) characterized by the slow decay of the flow field to its asymptotic values.

where $\mathbf{v} = [u, v]$ is the velocity vector, p is pressure, $\mathbf{e}_x$ is the unit vector associated with the X-axis, $\mathbf{x} = [x, y] \in \Omega$ is the position vector and $Re := U_\infty d/\nu$ is the Reynolds number in which ν is the kinematic viscosity (for simplicity, the fluid density is set equal to one). The symbol ":=" means "equal to by definition". Mathematical analysis of problem (1), which was initiated by Leray in the 1930s [42] and continued by Finn in the 1960s [21–23], is surveyed in the monograph by Galdi [30]. It reveals a number of interesting properties related to the behavior of the velocity field at large distances from the obstacle which is quite distinct from the corresponding time-dependent flows. More precisely, steady 2D flows described by (1) feature a "wake" region in the direction of the X-axis, cf. Fig. 1, in which the velocity field $\mathbf{v}$ approaches its asymptotic value $U_\infty \mathbf{e}_x$ much slower than outside this region, namely at the rate

$$|\mathbf{v}(\mathbf{x}) - U_\infty \mathbf{e}_x| = \mathcal{O}(|\mathbf{x}|^{-1/4-\epsilon}) \qquad \text{as } |\mathbf{x}| \to \infty, \tag{2}$$

where $\epsilon > 0$. Solutions of this type were referred to by Finn as "physically reasonable" (PR) [21] and have the additional property that to the leading order they have the same behavior at large distances as the solutions of the corresponding Oseen problem characterized by the same drag force [30], i.e.,

$$\mathbf{v}(\mathbf{x}) = U_\infty \mathbf{e}_x + \mathbf{F} \cdot \mathbf{E}(\mathbf{x}) + \mathbf{V}(\mathbf{x}) \qquad \text{as } |\mathbf{x}| \to \infty, \tag{3}$$

where $\mathbf{F} = [F_x\ F_y]^T$ is the hydrodynamic force acting on the obstacle A, $\mathbf{E}(\mathbf{x})$ is the fundamental solution tensor for the Oseen system

$$(U_\infty \mathbf{e}_x) \cdot \nabla \mathbf{u} + \nabla p - \frac{1}{Re} \Delta \mathbf{u} = \mathbf{0} \qquad \text{in } \Omega, \tag{4a}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{4b}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial A, \tag{4c}$$

$$\mathbf{u} \quad \to \mathbf{u}_\infty \quad \text{as } |\mathbf{x}| \to \infty, \tag{4d}$$

and the "remainder" $\mathbf{V}(\mathbf{x})$ satisfies the following asymptotic estimate

$$\mathbf{V}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1} \log^2 |\mathbf{x}|) \qquad \text{as } |\mathbf{x}| \to \infty. \tag{5}$$

In other words, at large distances from the obstacle the PR solutions are up to a rapidly vanishing correction indistinguishable from the Oseen flows exhibiting the same drag $\mathbf{F}$. Finn and Smith [22] showed that for small Reynolds numbers problem (1) has at least one solution that is physically reasonable. While it remains to be proven whether steady Navier–Stokes system (1) has solutions for *all* values of the Reynolds number, for now we will assume that at least one solution exists for all finite Reynolds numbers. In addition to making the numerical solution of problem (1) more challenging, the properties discussed above also complicate evaluation of the hydrodynamic forces [47].

The first calculation of a steady flow around a circular cylinder was carried out by Thom [51] for low Reynolds numbers ($Re = 10 - 20$) using the streamfunction–vorticity formulation. An interesting aspect of that research was the use of a conformal mapping. The simulations performed by Kawaguti [37] and by Apelt [3] for the Reynolds number up to 44 showed a linear growth of the vortex pair behind the cylinder with Re. Allen and Southwell [1] introduced upwind schemes to computational fluid dynamics when solving steady flows for Reynolds numbers up to 1000. Their solutions showed a trend of reduced recirculation length for the Reynolds number increasing from 10 to 100. The results of Hamielec and Raal [35] also indicated that the recirculation length decreased for Reynolds number larger than 50. We remark that, as discussed below, these results are now believed to be erroneous. Keller [38] and Takami [50] combined conformal mappings with finite-difference methods to solve steady flows around the cylinder for the Reynolds number up to 15, whereas a spectral method for the study of the stability of flows in unbounded domains was developed by Zebib [56]. These earlier investigations are reviewed in the historical survey by Fornberg [27]. Many numerical difficulties in solving system (1) stem from the fact that the unbounded domain Ω needed to be truncated to a finite computational box and it is not immediately obvious what boundary conditions must be prescribed on its boundary to ensure the solutions exhibit the correct asymptotic behavior given in (2) and (3).

The significance of the far-field boundary conditions was already recognized by Fornberg [25] who observed that the use of the free-stream values on the outer boundary of the computational domain produced large errors even for low Reynolds number. We note, however, that Fornberg considered the free-stream values for the streamfunction only while setting the vorticity equal to zero. In the numerical results of Fornberg [25] the length of the recirculation zone appears proportional to the Reynolds number. The recirculation width, however, exhibits different behavior depending on the Reynolds number: for $Re \lessapprox 300$ the width appears proportional to the square root of the Reynolds number; on the other hand, for $Re \gtrapprox 300$ the relation is linear. This behavior is also reflected in the different flow patterns observed in the two regimes with the flows obtained for $Re \lessapprox 300$ featuring a slender wake reminiscent of the Kirchhoff free-streamline solution ([40,43], see Fig. 2a) and those corresponding to $Re \gtrapprox 300$ characterized by a wider recirculation region more similar to the Prandtl–Batchelor limiting solution ([4], see Fig. 2b). Thus, although Fornberg's solutions [25] still represent the state-of-the-art in this field, they are rather inconclusive as regards the solution structure at large distances in the high-Reynolds number limit. There exist more recent results concerning two dimensional steady-state flows past obstacles, but they involve different configurations such as flows past arrays of obstacles as in [26,29], flows past obstacles in channels [48], or flows of stratified fluids [16].

The question of consistent boundary condition imposed on the boundaries of the computational domain was recently taken up by Bönisch et al. [6–8]. As will be discussed below, they devised an adaptive approach in which the corrections to the free-stream are consistent with (3) and depend on the force experienced by the obstacle. More recent attempts at solving problem (1), although not necessarily focusing on obtaining solutions in the high-Re limit, include [54,17,31] with study [31] containing certain similar ideas to those investigated here. In the context of time-dependent flows, the question of surrogate boundary conditions on truncated domains was recently also addressed in [19].

The main contribution of our study is development of a spectrally-accurate solution method based on the streamfunction–vorticity formulation ensuring that asymptotic condition (3) is satisfied. As discussed below, the key technical difficulty in this approach is the resolution of the singularity appearing at infinity in the streamfunction field which is achieved through a suitable