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A general framework for computing the turbulence structure tensors

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ABSTRACT

Good measures of the turbulence structure are important for turbulence modeling, flow diagnostics and analysis. Structure information is complementary to the componentality anisotropy that the Reynolds stress tensor carries, and because structures extend in space, structure information is inherently non-local. Given access to instantaneous snapshots of a turbulence field or two-point statistical correlations, one can extract the structural features of the turbulence. However, this process tends to be computationally expensive and cumbersome. Therefore, one-point statistical measures of the structural characteristics of turbulence are desirable. The *turbulence structure tensors* are one-point statistical descriptors of the non-local characteristics of the turbulence structure and form the mathematical framework for constructing Structure-Based Models (SBM) of turbulence. Despite the promise held by SBM, the tensors have so far been available only in a small number of DNS databases of rather simple canonical flows. This inhibits further SBM development and discourages the use of the tensors for flow analysis and diagnostics. The lack of a clear numerical recipe for computing the tensors in complex domains is one the reasons for the scarce reporting of the structure tensors in DNS databases. In particular, the imposition of proper boundary conditions in complex geometries is non-trivial. In this work, we provide for the first a time a rigorous and well-documented description of a mathematical and computational framework that can be used for the calculation of the structure tensors in arbitrary turbulent flow configurations.

1. Introduction

1.1. Background

Far from being equivalent to white noise, the turbulent motion of fluids is organized in the form of coherent structures, often given the label 'eddies'. In high Reynolds number flows, the size of the turbulence eddies can span several orders of magnitude. In these flows, the small-scale structure is thought to be effectively shielded from external forcing and thus exhibits a significant degree of isotropy as a result. It is further assumed that the role of the smaller eddies is primarily to dissipate the turbulence energy. The larger energy-containing structures, on the other hand, are both shaped by and play a role in determining the response of turbulence to external deformation. They are dynamically active. The footprint of these large energy-containing turbulence eddies is reflected in the turbulence statistics. Quantitative measures of turbulence structure are easily constructed using two-point correlations, but such descriptions tend to be rather costly and impractical for engineering application, which relies heavily on one-point formulations. Hence, one-point measures of turbulence structure are needed. Kassinos and Reynolds [15] were the first to develop a comprehensive one-point mathematical formulation that can be used to quantify different aspects of the energy containing turbulence structures. They proceeded to propose the use of the one-point turbulence structure tensors in turbulence modeling and for flow diagnostics, which they described in [7,15,16]. In this regard, they showed that it is possible for two turbulence fields to share the same *componentality* state, i.e. to have the same Reynolds stress tensor values, but yet have different underlying turbulence structure. Differences in the turbulence structure, although undetectable through the componentality information, lead to different dynamic behavior of the turbulence, for example in response to external deformation. Hence, a complete one-point description of the turbulence requires the information contained in the structure tensors. Namely, the structure *dimensionality* D_{ij} gives information about the directions of independence in the turbulence, the structure *circulicity* F_{ij} gives information on the large scale circulation in the flow, and the *inhomogeneity* C_{ij} gives the degree of inhomogeneity of the

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turbulence. The third-rank *stropholysis* Q_{ijk} becomes important when mean rotation breaks the reflectional symmetry of the turbulence. Exact definitions of these tensors are given in the next section.

One-point turbulence models that use only the Reynolds stresses and the turbulence scales to characterize the turbulence are fundamentally incomplete [15]. This applies to both simple eddy-viscosity closures and to Reynolds Stress Transport (RST) models and it is particularly problematic when the mean deformation includes strong mean or frame rotation. For example, in this case, the dynamic response of nearly isotropic turbulence is very different from that of turbulence with strongly organized two-dimensional structures and turbulence models should be able to distinguish between the two. Without ad hoc modifications, most turbulence closures, however, fail to do that because they are insensitive to the structural characteristics of the turbulence. Furthermore, turbulence models should incorporate the breaking of reflectional symmetry whenever mean or frame rotation can dynamically affect the flow (flow through axisymmetric diffuser or nozzle with swirl, flow through turbomachinery). These aforementioned effects are nonlocal in nature, yet they can be addressed via the one-point structure tensors, which is the main feature of the tensors that makes them particularly attractive in engineering practice.

Structure-Based turbulence Models (SBMs) [11,13,15,16,20] are a class of turbulence models that make use of the one-point turbulence tensors. SBMs hold promise for resolving some of the limitations described above. However, an obstacle in the further development of structure-based models has been the relatively scarce availability of data from simulations and experiments that could be used for model validation. On one hand, the one-point structure tensors are not easily available from experiments. Hence, one normally has to turn to direct (DNS) or large eddy simulations (LES) for obtaining data on the tensors. Even in this case, however, the specification of proper boundary conditions for the computation of the structure tensors has so far been considered only in the simplest geometries, e.g. fully-developed channel flow and free shear flows [16]. The underlying ambiguity over how one can compute the tensors in complex domains has discouraged the more widespread inclusion of the tensors in turbulence databases. This in turn has hurt the development of structure-based models and also prevented the more widespread use of the tensors as flow diagnostics. As SBM testing and validation progresses to complex flow configurations this limitation becomes more pressing.

The purpose of this work is to present a clear framework for the numerical computation of the structure tensors in arbitrarily complex geometries using DNS or LES data. We believe that this contribution will encourage the inclusion of the structure tensors in DNS databases, thus accelerating the development of structure-based models and encouraging the use of one-point structure tensors as flow diagnostic tools.

1.2. Definition of the structure tensors

The structure tensors are determined through the turbulent stream vector ψ'_i, defined by the equations [16]

$$u'_i = \epsilon_{ijk}\psi'_{k,j} \qquad \psi'_{k,k} = 0 \qquad \psi'_{i,kk} = -\omega'_i, \tag{1}$$

where u'_i and ω'_i are the fluctuating velocity and vorticity components, and ϵ_{ijk} is the Levi–Civita alternating tensor. To complete the stream vector definition suitable boundary conditions must be supplied. Hereafter, a comma followed by an index denotes partial differentiation with respect to the implied coordinate direction. The Einstein summation convention is implied on repeated Roman, but not on Greek indices. Note that ψ'_i satisfies a Poisson equation and hence carries non-local information. As will be shown, the divergence-free condition on ψ'_i is important for the physical meaning of the resulting structure tensors. The focus of this paper is a general strategy for solving (1) in complex domains, thus making possible the computation of the structure tensors in practical flow configurations.

Expressing the definition of the Reynolds stresses in terms of the fluctuating stream vector,

$$R_{ij} = \overline{u'_i u'_j} = \epsilon_{ipq}\epsilon_{jrs}\overline{\psi'_{q,p}\psi'_{s,r}}, \tag{2}$$

and using the identity

$$\epsilon_{ipq}\epsilon_{jrs} = \det\begin{pmatrix} \delta_{ij} & \delta_{ir} & \delta_{is} \\ \delta_{pj} & \delta_{pr} & \delta_{ps} \\ \delta_{qj} & \delta_{qr} & \delta_{qs} \end{pmatrix}, \tag{3}$$

leads to the constitutive relation

$$R_{ij} + D_{ij} + F_{ij} - (C_{ij} + C_{ji}) = \delta_{ij}q^2, \tag{4}$$

where $q^2 = R_{kk} = 2k$ is twice the turbulent kinetic energy. Based on this equation, the second-rank structure tensors are defined as

$$\text{Componentality} \quad R_{ij} = \overline{u'_i u'_j} \qquad r_{ij} = R_{ij}/R_{kk} \qquad \tilde{r}_{ij} = r_{ij} - \delta_{ij}/3, \tag{5a}$$

$$\text{Dimensionality} \quad D_{ij} = \overline{\psi'_{k,i}\psi'_{k,j}} \qquad d_{ij} = D_{ij}/D_{kk} \qquad \tilde{d}_{ij} = d_{ij} - \delta_{ij}/3, \tag{5b}$$

$$\text{Circulicity} \quad F_{ij} = \overline{\psi'_{i,k}\psi'_{j,k}} \qquad f_{ij} = F_{ij}/F_{kk} \qquad \tilde{f}_{ij} = f_{ij} - \delta_{ij}/3, \tag{5c}$$

$$\text{Inhomogeneity} \quad C_{ij} = \overline{\psi'_{i,k}\psi'_{k,j}} \qquad c_{ij} = C_{ij}/D_{kk} \qquad \tilde{c}_{ij} = c_{ij} - c_{kk}\delta_{ij}/3. \tag{5d}$$

Unlike the other structure tensors, the inhomogeneity C_{ij} is not positive semi-definite and thus the trace $C_{kk} = D_{kk} - R_{kk}$ can be negative or even zero. For this reason, C_{ij} is normalized in terms of the trace $D_{kk} = F_{kk}$. Another possibility would have been to normalize all structure tensors with the trace R_{kk}, but this choice is ill-defined on solid boundaries, where R_{kk} is zero. On the contrary, D_{kk} is non-zero at the walls and proves to be the most meaningful choice for normalizing all the structure tensors.

A detailed discussion of the physical meaning of each structure tensor is given in [16], but the key features are recounted here. While the structure tensors carry complementary information, the constitutive equation provides a linear dependence among them, thus any one of the tensors can be reconstructed if the rest are known. The *componentality* R_{ij} (the Reynolds stress tensor) gives information about which components of the fluctuating velocity are more energetic. The *dimensionality* D_{ij} carries information about the directions of independence of the turbulence. This can be easily seen based on the definition of D_{ij}, since gradients of the stream vector tend to vanish along directions of strong structure elongation and tend to be strongest along directions in which short structures are stacked. The *circulicity* F_{ij} defines the directions with large-scale circulation concentrated around them. Finally, the *inhomogeneity* C_{ij} gives the directions of inhomogeneity of the turbulence. In fact, the inhomogeneity tensor vanishes identically in homogeneous flows, as can be shown by recasting the inhomogeneity definition into the form

$$C_{ij} = (\overline{\psi'_i\psi'_{k,j}})_{,k} - \overline{\psi'_i\psi'_{k,kj}}. \tag{6}$$

Here, the first term is zero only in homogeneous flows, while the second term is always zero due to the specific choice $\psi_{k,k} = 0$. The inhomogeneity is significant near solid boundaries and relaxes to zero far away from them. At intermediate distances form the wall, the magnitude of C_{ij} becomes small compared to that of the other structure tensors. Since little is known on how to model C_{ij} in general flows, structure-based turbulence models, such as the Algebraic Structure-Based Model (ASBM) [2,14,17,22], are based on the *homogenized* tensors. These are obtained by absorbing C_{ij} inside D_{ij} and F_{ij},