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Capturing matched layer at absorbing boundary with finite volume schemes

Ching Y. Loh [*,1]

Beijing Computational Sciences Research Center, 3 Heqing Rd., Haidian District, Beijing 100084, China

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ABSTRACT

In compressible flow computations, an important treatment for absorbing boundary condition (ABC) is to create a matched layer at the boundary, as in the well-known PML (perfectly matched layer) method. In the present paper, it is shown that with cell-centered finite volume (FV) schemes, the matched layer can be captured directly as a discontinuity across the absorbing boundary, rendering an extremely simple yet robust ABC. The new ABC is inherently embedded in FV schemes of cell-centered type, and often associated with the captured matched layer, which serves to match the flow variables across the boundary. It has been used empirically for years and was found to have no direct relation to any existing nonreflecting boundary condition (NRBC). Instead, it is attributed to the shock-capturing capability of the FV scheme, as well as a nonreflecting (NR) observation that for any scheme, no spurious reflection is generated at any *interior* point of the domain. A Fourier analysis with plane waves is employed to the local Euler solution to justify the NR observation and the ABC is consequently established.

The ABC performs perfectly with zero reflection in one dimensional space. In multi-dimensional spaces, the phase error and reflection due to discretization are discussed. With appropriate grid resolution at the boundary, one can always suppress the spurious reflection to any designated level. Several non-trivial numerical examples in one and multi-dimensional spaces are tested to demonstrate its robustness.

1. Introduction

Nonreflecting boundary conditions (NRBCs) play a key role in fluid dynamics computations while remaining as a challenging topic in the current research areas in engineering and applied mathematics. Spurious reflections resulting from an inappropriate numerical boundary condition contaminate the flow field and may eventually spoil the entire computed flow. There have been a huge number of literatures dealing with the topic of NRBC. Herein, we focus only on the related issues of the absorbing boundary condition (ABC).

For a boundary with flow conditions specified, an ABC plays a dual role: to enforce the given boundary conditions and to absorb the wave or disturbance propagating from the domain interior. Usually, there are two ways to absorb the outgoing waves and to avoid spurious reflections at the boundary. In one way, the given flow boundary condition is modified to an admissible numerical boundary condition, as in the characteristics-based NRBC (cf. e.g., [2]). In the other way, a matched layer (or absorbing layer) is artificially created at the boundary, which smoothly matches the inconsistent flow data across the boundary. A typical work along this track is the recent perfectly matched layer (PML) method (cf. e.g., [3–5]).

In recent years, another interesting ABC for finite volume (FV) schemes was discovered empirically, e.g., Leveque [6] and Loh et al. [7–10]. When the flow conditions prescribed at the ghost cell centers (GCCs) do not match the flow within the domain, a matched layer similar to the one in the PML method is *automatically* captured at the boundary to do the matching, saving the work of creating a matched layer or matching the flow data. The ABC is hence extremely simple yet robust. The purpose of the present paper is to examine and establish this new ABC for FV schemes.

Despite its attractive advantages, investigations show that the new ABC does not seem to directly relate to any existing NRBC, but should be credited to the shock-capturing capability of FV schemes, as well as a well-accepted empirical observation that no spurious reflection occurs at any *interior* point of the domain. For brevity, hereafter, this observation will be called the nonreflecting observation or NR observation.

The contents are arranged as follows. First of all, Sections 2 and 3 present the theoretical work to prove the NR observation. In Section 2, We begin with the introduction of a simple but

* Permanent address: 2237 Gunstock Dr., Stone Mountain, GA 30087, USA. Tel.: +1 905 513 8058.
[1] Visiting professor.
E-mail addresses: ken.loh09@gmail.com, kenloh@163.com

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unconventional concept of absorbing. Based on this concept, Fourier analysis is conducted to the local Euler solution in Section 3 to justify the NR observation. If, by his/her own experience, the reader would accept the NR observation for granted, Sections 2 and 3 may be temporarily skipped, as they are tedious and require concepts in real analysis.

Section 4 shows that, due to a boundary treatment in cell-centered FV schemes, the NR observation can be extended and applied to the boundary itself. The new ABC is then established based on the shock-capturing capability of the FV schemes. With a one dimensional example, the section illustrates how the ABC works and the automatic capturing/creation of the associated absorbing layer (matched layer).

The ABC was found working perfectly in one dimensional space with zero reflection, but in multi-dimensional computations, lack of sufficient grid resolution on the boundary may still introduce spurious reflection. Section 5 is devoted to the phase error analysis at a boundary element (a line segment or a surface element) in multi-dimensional spaces, and the discussion of reflection coefficient. So that the spurious reflection can be suppressed to any designated level by choosing an appropriate grid resolution at the boundary. In Section 6 the new ABC is tested in several non-trivial examples in multi-dimensional spaces. Finally, the paper is concluded with remarks in Section 7.

Throughout the paper, $\mathbf{V}(\mathbf{x},t) = (\rho, u, v, w, p)^T$ is employed to represent the primitive flow variables in the solution of the three dimensional Euler equations, where $\mathbf{x}$, t, u, v, w, ρ, p are respectively the coordinates (x,y,z), time, the three velocity components, density, and pressure. $\mathbf{W}(\mathbf{x},t) = (\rho, \rho u, \rho v, \rho w, \rho e)^T$ represents the conservative flow variables, with the energy

$$e = \frac{p}{\rho(\gamma-1)} + \frac{1}{2}(u^2+v^2+w^2), \quad \gamma = 1.4.$$

$\mathbf{U}(\mathbf{x},t)$ represents a numerical solution or an approximation of $\mathbf{V}(\mathbf{x},t)$.

2. Simple concept of absorbing

Our past investigations show that the new ABC cannot be inferred from any existing NRBC. For further examination, a simple but unconventional concept of absorbing is put forward as our definition for absorbing. It is based on the plane waves in the theory of linear partial differential equations [1], and is introduced below step by step in a logical way by mathematical analysis. The term "local Euler solution" refers to the local analytical solution of the Euler equations in a small neighborhood of an *interior* point in the domain. No numerical solution is involved yet at this stage.

All the following conceptual discussions are focused in an infinitely small neighborhood, Ω, of a point, x_0 (or $\mathbf{x}_0$) in one or multi-dimensional space.

1. First, consider a simple scalar sine wave defined on the real axis:

$$f(x) = a\sin(bx+c), \quad -\infty < x < \infty, \tag{1}$$

where a, b, c are given real constants. The sine wave has the simplest form of a wave, and is completely determined by its amplitude, a, and its phase $bx+c$. In Ω, no matter how x approaches x_0 (from left or right),

$$a = const., \quad \lim_{x\to x_0\pm 0}(bx+c) = bx_0 + c, \tag{2}$$

the sine wave, i.e., its amplitude and phase, always remains intact across x_0, or no distortion occurs at x_0. *No distortion means no spurious reflection.* The sine wave is thus said to be *absorbed* at x_0 (in either direction). Here, "intact", "no distortion", "absorbed", and "no spurious reflection" are synonymous terms. This forms the basic concept of absorbing for a sine wave at a point x_0.

2. Next, the above concept is extended to the most general case of a vector plane wave (cf. [1], or Appendix B) in three-dimensional space:

$$\mathbf{V}(\mathbf{k},\mathbf{x},t) = \widetilde{\mathbf{V}}(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{3}$$

where $i=\sqrt{-1}$, $\mathbf{x} = (x,y,z)$ represents the position vector of a point in three-dimensional space, $\mathbf{k} = (k_x,k_y,k_z)$ is the given wave number vector, with its real components, k_x, k_y, and k_z. The amplitude vector, $\widetilde{\mathbf{V}}(\mathbf{k})$, and the angular frequency, ω are functions of $\mathbf{k}$ only (cf. [1], or Appendix B). They are constant with $\mathbf{k}$ given. t is time, when the wave propagation pattern in space is investigated, time is fixed at $t=t_0$. Like the simple sine wave, $\mathbf{V}(\mathbf{k},\mathbf{x},t)$ is completely determined by its amplitude $\widetilde{\mathbf{V}}(\mathbf{k})$ and phase, $\theta = \mathbf{k}\cdot\mathbf{x} - \omega t$. Similar to (2),

$$\widetilde{\mathbf{V}}(\mathbf{k}) = const., \quad \lim_{\mathbf{x}\to\mathbf{x}_0}\theta = \mathbf{k}\cdot\mathbf{x}_0 - \omega t_0, \tag{4}$$

the vector plane wave, $\mathbf{V}(\mathbf{k},\mathbf{x},t_0)$ along with its amplitude and phase, remains intact or is absorbed at any point $\mathbf{x}_0$, when $\mathbf{x}$ approaches the point in any direction or path in its small neighborhood, Ω.

This is the basic concept of absorbing for a single vector plane wave.

3. Finally, let $\mathbf{V}(\mathbf{x},t_0)$ be a superposition of plane waves $\mathbf{V}(\mathbf{k},\mathbf{x},t_0)$ of different wave numbers $\mathbf{k}$, for example:

$$\mathbf{V}(\mathbf{x},t_0) = \sum_{n=1}^{N}\mathbf{V}(\mathbf{k}_n,\mathbf{x},t_0), \quad \text{or}$$
$$\mathbf{V}(\mathbf{x},t_0) = \iiint \mathbf{V}(\mathbf{k},\mathbf{x},t_0)\cdot dk_x dk_y dk_z, \tag{5}$$

where N is the number of plane waves, and the integral is carried over a domain in the three-dimensional wave number space. As each of these plane waves remains intact across a point $\mathbf{x}_0$, their superposition, $\mathbf{V}(\mathbf{x},t_0)$, is also regarded as remaining intact or being absorbed at $\mathbf{x}_0$.

In summary, the concept of absorbing can be stated as follows: *If in a (infinitely) small neighborhood of a point, $\mathbf{x}_0$, the local Euler solution can be expressed as a superposition of plane waves, it is absorbed at this point.*

It will be used to determine theoretically whether a local Euler solution is absorbed at a point.

3. Fourier analysis of the local Euler solution

As in Section 2, the proof in this section is still theoretical, using the concepts in real analysis. No numerical treatment is involved yet. It will be shown *theoretically* that, at any *interior* point of the computational domain, the local Euler solution, $\mathbf{V}(\mathbf{x},t_0)$, and its numerical approximation, $\mathbf{U}(\mathbf{x},t_0)$, can be expressed as a superposition of plane waves. Hence they are absorbed, and the NR observation is proved. Here, by *interior* point, we mean a point completely lying within the domain. Points at the domain boundary or internal boundaries must be excluded, as at least a small neighborhood of this point is required lying completely within the domain.

3.1. Fourier analysis of the local Euler solution

To do so, Fourier analysis is conducted to the local Euler solution within an infinitely small neighborhood, Ω, of the given interior point, $\mathbf{x}_0 = (x_0,y_0,z_0)$, in the following steps.