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The iterative transformation method for the Sakiadis problem

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ABSTRACT

In a transformation method the numerical solution of a given boundary value problem is obtained by solving one or more related initial value problems. This paper is concerned with the application of the iterative transformation method to the Sakiadis problem. This method is an extension of the Töpfer's non-iterative algorithm developed as a simple way to solve the celebrated Blasius problem. As shown by this author (Fazio, 1997) the method provides a simple numerical test for the existence and uniqueness of solutions. Here we show how the method can be applied to problems with a homogeneous boundary conditions at infinity and in particular we solve the Sakiadis problem of boundary layer theory. Moreover, we show how to couple our method with Newton's root-finder. The obtained numerical results compare well with those available in literature. The main aim here is that any method developed for the Blasius, or the Sakiadis, problem might be extended to more challenging or interesting problems. In this context, the iterative transformation method has been recently applied to compute the normal and reverse flow solutions of Stewartson for the Falkner–Skan model (Fazio, 2013).

1. Introduction

In a transformation method the numerical solution of a given boundary value problem is obtained by solving one or more related initial value problems (IVPs). In this context the classical example is the Blasius problem of boundary layer theory. In the Blasius problem the governing differential equation and the two initial conditions are invariant under the scaling group of transformations

$$f^* = \lambda^{-\alpha} f, \quad \eta^* = \lambda^{\alpha}\eta, \tag{1.1}$$

where λ is the group parameter and $\alpha \neq 0$. Moreover, the non-homogeneous asymptotic boundary condition is not invariant with respect to (1.1). This kind of invariance was used by Töpfer [27] to define a non-iterative transformation method (ITM) for the Blasius problem by transforming the boundary conditions to initial conditions and rescaling the obtained numerical solution.

This paper is concerned with the application of an ITM to the Sakiadis problem. The main aim here is that any method developed for the Blasius, or the Sakiadis, problem might be extended to more challenging or interesting problems. In this context, the iterative transformation method has been recently applied to compute the normal and reverse flow solutions of Stewartson [25,26] for the Falkner–Skan model [13].

The Sakiadis problem is a variant of Blasius problem that cannot be solved by a non-ITM. In fact, one of the initial conditions is not invariant and the asymptotic boundary condition, being homogeneous, is invariant with respect to the scaling transformations (1.1). Therefore, as noted by Na [21, pp. 160-164], it is not possible to rescale an initial value solution to the given asymptotic boundary condition. Moreover, the non-ITM cannot be applied when the governing differential equation is not invariant with respect to a scaling group of point transformations. To overcome this drawback the ITM was defined in [8,9] for the numerical solution of the Falkner–Skan model and of other problems in boundary layer theory.

Here we show how the ITM can be applied to problems with a homogeneous boundary conditions at infinity. Moreover, we show how to couple our method with the Newton's root-finder. This ITM has been applied to several problems of interest: free boundary problems [5,10], a moving boundary hyperbolic problem [7], the Falkner–Skan equation in [8,9,13], one-dimensional parabolic moving boundary problems [11,14], two variants of the Blasius problem [12], namely: a boundary layer problem over moving plates, studied first by Klemp and Acrivos [19], and a boundary layer problem with slip boundary condition, that has found application to the study of gas and liquid flows at the micro-scale regime [4,20], a parabolic problem on unbounded domain [15]. Furthermore, as shown in [10], the ITM provides a simple numerical test for the existence and uniqueness of solutions.

2. Blasius and Sakiadis problems

Within boundary-layer theory, the model describing the steady plane flow of a fluid past a thin plate, is given by

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$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{2.1}$$

where the governing differential equations, namely conservation of mass and momentum, are the steady-state 2D Navier–Stokes equations under the boundary layer approximations: $u \gg v$ and the flow has a very thin layer attached to the plate, u and v are the velocity components of the fluid in the x and y direction, and ν is the viscosity of the fluid. The boundary conditions for the velocity field are

$$u(x,0)=v(x,0)=0, \quad u(0,y)=U_\infty, \qquad u(x,y)\to U_\infty \quad \text{as} \quad y\to\infty, \tag{2.2}$$

for the Blasius flat plate flow problem [2], where U_∞ is the main-stream velocity, and

$$u(x,0)=U_p, \quad v(x,0)=0, \qquad u(x,y)\to 0 \quad \text{as} \quad y\to\infty, \tag{2.3}$$

for the classical Sakiadis flat plate flow problem [23,24], where U_p is the plate velocity, respectively. The boundary conditions at $y=0$ are based on the assumption that neither slip nor mass transfer are permitted at the plate whereas the remaining boundary condition means that the velocity v tends to the main-stream velocity U_∞ asymptotically or gives the prescribed velocity of the plate U_p. Introducing a similarity variable η and a dimensionless stream function $f(\eta)$ as

$$\eta = y\sqrt{\frac{U}{\nu x}}, \quad u = U\frac{df}{d\eta}, \quad v=\frac{1}{2}\sqrt{\frac{U\nu}{x}}\left(\eta\frac{df}{d\eta}-f\right), \tag{2.4}$$

we have

$$\frac{\partial u}{\partial x}=-\frac{U}{2}\frac{\eta}{x}\frac{d^2f}{d\eta^2}, \quad \frac{\partial v}{\partial y}=\frac{U}{2}\frac{\eta}{x}\frac{d^2f}{d\eta^2} \tag{2.5}$$

and the equation of continuity, the first equation in (2.1), is satisfied identically. On the other hand, we get

$$\frac{\partial u}{\partial y}=U\frac{d^2f}{d\eta^2}\sqrt{\frac{U}{\nu x}}, \quad \frac{\partial^2 u}{\partial y^2}=\frac{U^2}{\nu x}\frac{d^3f}{d\eta^3}. \tag{2.6}$$

Let us notice that, in the above equations $U=U_\infty$ represents Blasius flow, whereas $U=U_p$ indicates Sakiadis flow, respectively.

By inserting these expressions into the momentum equation, the second equation in (2.1), we get

$$\frac{d^3f}{d\eta^3}+\frac{1}{2}f\frac{d^2f}{d\eta^2}=0, \tag{2.7}$$

to be considered along with the transformed boundary conditions

$$f(0)=\frac{df}{d\eta}(0)=0, \quad \frac{df}{d\eta}(\eta)\to 1 \quad \text{as} \quad \eta\to\infty,$$

for the Blasius flow, and

$$f(0)=0, \quad \frac{df}{d\eta}(0)=1, \quad \frac{df}{d\eta}(\eta)\to 0 \quad \text{as} \quad \eta\to\infty,$$

for the Sakiadis flow, respectively.

Blasius main interest was to compute the value of the velocity gradient at the plate (the wall shear or skin friction coefficient):

$$\lambda=\frac{d^2f}{d\eta^2}(0).$$

To compute this value, Blasius used a formal series solution around $\eta=0$ and an asymptotic expansions for large values of η, adjusting the constant λ so as to connect both expansions in a middle region. In this way, Blasius obtained the (erroneous) bounds $0.3315 < \lambda < 0.33175$.

A few years later, Töpfer [27] revised the work by Blasius and solved numerically the Blasius problem, using a non-ITM. He then arrived, without detailing his computations, at the value $\lambda \approx 0.33206$, contradicting the bounds reported by Blasius.

Indeed, Töpfer solved the IVP for the Blasius equation once. At large but finite η_j^*, ordered so that $\eta_j^* < \eta_{j+1}^*$, he computed the corresponding scaling parameter λ_j. If two subsequent values of λ_j agree within a specified accuracy, then λ is approximately equal to the common value of the λ_j, otherwise, he marched to a larger value of η^* and tried again. Using the classical fourth order Runge–Kutta method, as given by Butcher [3, p. 166], and a grid step $\Delta\eta^*=0.1$ Töpfer was able, only by hand computations, to determine λ with an error less than 10^{-5}. To this end he used the two *truncated boundaries* $\eta_1^*=4$ and $\eta_2^*=6$. For the sake of simplicity we follow Töpfer and apply some preliminary computational tests to find a suitable value for the truncated boundary.

Sakiadis studied the behavior of boundary layer flow, due to a moving flat plate immersed in an otherwise quiescent fluid, [23,24]. He found that the wall shear is about 34% higher for the Sakiadis flow compared to the Blasius case. Later, Tsou and Goldstein [28] made an experimental and theoretical treatment of Sakiadis problem to prove that such a flow is physically realizable.

3. Extension of Töpfer algorithm: the ITM

Within this section we explain how it is possible to extend Töpfer algorithm to the Sakiadis problem, that we rewrite here for the reader convenience

$$\frac{d^3f}{d\eta^3}+\frac{1}{2}f\frac{d^2f}{d\eta^2}=0, \qquad f(0)=0, \quad \frac{df}{d\eta}(0)=1, \quad \frac{df}{d\eta}(\eta)\to 0 \quad \text{as} \quad \eta\to\infty. \tag{3.1}$$

In order to define the ITM we introduce the extended problem

$$\frac{d^3f}{d\eta^3}+\frac{1}{2}f\frac{d^2f}{d\eta^2}=0, \qquad f(0)=0, \quad \frac{df}{d\eta}(0)=h^{1/2}, \quad \frac{df}{d\eta}(\eta)\to 1-h^{1/2} \quad \text{as} \quad \eta\to\infty. \tag{3.2}$$

In (3.2), the governing differential equation and the two initial conditions are invariant, the asymptotic boundary condition is not invariant, with respect to the extended scaling group

$$f^*=\lambda f, \quad \eta^*=\lambda^{-1}\eta, \quad h^*=\lambda^4 h. \tag{3.3}$$

Moreover, it is worth noticing that the extended problem (3.2) reduces to the Sakiadis problem (3.1) for $h=1$. This means that in order to find a solution of the Sakiadis problem we have to find a zero of the so-called *transformation function*

$$\Gamma(h^*)=\lambda^{-4}h^*-1, \tag{3.4}$$

where the group parameter λ is defined with the formula

$$\lambda=\left[\frac{df^*}{d\eta^*}(\eta_\infty^*)+h^{*1/2}\right]^{1/2}, \tag{3.5}$$

and to this end we can use a root-finder method.

Let us notice that λ and the transformation function are defined implicitly by the solution of the IVP

$$\frac{d^3f^*}{d\eta^{*3}}+\frac{1}{2}f^*\frac{d^2f^*}{d\eta^{*2}}=0, \qquad f^*(0)=0, \quad \frac{df^*}{d\eta^*}(0)=h^{*1/2}, \quad \frac{d^2f^*}{d\eta^{*2}}(0)=\pm 1. \tag{3.6}$$