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A space–time adaptive discontinuous Galerkin scheme

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ABSTRACT

A discontinuous Galerkin scheme for unsteady fluid flows is described that allows a very high level of adaptive control in the space–time domain. The scheme is based on an explicit space–time predictor, which operates locally and takes the time evolution of the data within the grid cell into account. The predictor establishes a local space–time approximate solution in a whole space–time grid cell. This enables a time-consistent local time-stepping, by which the approximate solution is advanced in time in every grid cell with its own time step, only restricted by the local explicit stability condition. The coupling of the grid cells is solely accomplished by the corrector which is determined by the numerical fluxes. The considered discontinuous Galerkin scheme allows non-conforming meshes, together with *p*-adaptivity in 3 dimensions and *h*/*p*-adaptivity in 2 dimensions. Hence, we combine in this scheme all the flexibility that the discontinuous Galerkin approach provides. In this work, we investigate the combination of the local time-stepping with *h*- and *p*-adaptivity. Complex unsteady flow problems are presented to demonstrate the advantages of such an adaptive framework for simulations with strongly varying resolution requirements, e.g. shock waves, boundary layers or turbulence.

1. Introduction

An important advantage of the discontinuous Galerkin (DG) approach consists of its high flexibility with respect to the spatial approximation: The approximation of the solution is represented by a piecewise polynomial and the degree of the local polynomials determine the order of accuracy in space. Hence, increasing the order can be achieved through a simple enrichment of the local basis. This can also be handled locally in regions where we aim a higher resolution. Geometrical flexibility is also an important feature of the DG method since it can be formulated on general unstructured grids with non-conforming meshes to handle complex geometries. The DG method is thus an ideal candidate, for which general adaptive strategies as local grid refinement (*h*-refinement) or local choice of the order of accuracy (*p*-adaptivity) can be applied, see, e.g., [25]. Due to this distinguished versatile flexibility the DG method is particularly suited for multi-scale problems which require small regions of high resolution while other parts can be discretized uniformly.

Adaptive concepts for discontinuous Galerkin schemes have been successfully applied, e.g., by Burgess and Mavriplis [7], Kopera and Giraldo [26] or Hartmann and Houston [22,23]. All these approaches based on DG schemes with implicit or semi-implicit time integration. As we are particularly interested in unsteady flow problems, the adaptivity of the time approximation is an additional issue and is a focal point in this paper. Especially for wave dominated problems like acoustics and propagating shock waves, the time accuracy has to be preserved. Regions with local refinement and unsteady phenomena need small time steps. Allowing large time steps for a transient problem would lead to numerical diffusion and to loss of information. For an explicit scheme, a global time-stepping approach can cause inefficiency, since the smallest time step forced by small sized grid cells dominates the whole simulation. To overcome this inefficiency, we employ an explicit time approximation with local time-stepping, which allows each grid cell to take the maximum locally stable time step. The local time-stepping enables a powerful adaptation framework, which also takes the coupling of spatial and temporal scales into account since any local *h*-refinement or *p*-enrichment in space inherently implies an adaptation of the time step as well. Together with the explicit local time-stepping approach, the artificial viscosity based shock capturing introduced by Persson and Peraire for an implicit DG scheme [35] now also becomes an attractive method to robustly resolve shock waves.

In the following we describe an explicit discontinuous Galerkin scheme that incorporates the following properties:

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- high order accurate spatial approximations on general unstructured grids with triangles, quadrilaterals, tetrahedra, hexahedra, pyramids, prisms,
- arbitrary high order in space and time,
- non-conforming meshes in 2 and 3 dimensions,
- fully conservative,
- shock capturing with artificial viscosity,
- local time-stepping with the time step from the local stability constraint,
- h/p-adaptivity in 2 dimensions and p-adaptivity in 3 dimensions.

Different building blocks of our DG approach have been constructed and already been published in a series of papers [30,15,16,19]. The scope of this work is the investigation of the combination and interaction of explicit local time-stepping, artificial viscosity based shock capturing and h/p-adaptivity. To demonstrate the high potential of the adaptivity framework, simulation results of various complex unsteady flow problems are presented.

The paper is organized as follows. We first describe the main components of the space approximation within the semi-discrete approach in Section 2. The time consistent local time-stepping method and the determination of the time step are introduced in Section 3 to obtain the fully discrete formulation. The adaptivity framework (h- and p-adaptivity) and its implementation with respect to the local time-stepping scheme is discussed in Section 4. In Section 4 we also provide a brief overview of the artificial viscosity based shock capturing as part of the whole adaptation strategy. In Section 5, we focus on the simulation results and present different applications of the space–time-adaptive framework for two- and three-dimensional complex flows and also address the parallel performance and dynamic load balancing attributes of the implementation. Conclusions and final remarks are given in Section 6.

2. An explicit discontinuous Galerkin method in the space–time domain

2.1. Approximation in space

For simplicity we restrict the derivations to a scalar advection–diffusion equation, which is formulated in conservation form as

$$u_t + \vec{\nabla}\cdot\vec{f}^a(u) = \vec{\nabla}\cdot\vec{f}^d(u,\vec{\nabla}u). \tag{2.1}$$

Here, $u = u(\vec{x},t)$ denotes the conserved solution variable, $\vec{f}^a$ is the advection flux and $\vec{f}^d$ the diffusion flux. The diffusion in the flux formulation may be rewritten as

$$\vec{\nabla}\cdot\vec{f}^d(u,\vec{\nabla}u) = \vec{\nabla}\cdot\Big(\mu(u)\vec{\nabla}u\Big), \tag{2.2}$$

where $\mu = \mu(u)$ is the non-linear diffusion coefficient.

First, we consider the discretization in space. The spatial domain Ω is subdivided into non-overlapping spatial grid cells Q_i with surfaces ∂Q_i. The variational formulation of the advection–diffusion Eq. (2.1) is obtained by multiplying the equation by a test function $\phi = \phi(\vec{x})$ and by integration over a grid cell Q_i:

$$\int_{Q_i}\Big(u_t + \vec{\nabla}\cdot(\vec{f}^a(u) - \vec{f}^d(u,\vec{\nabla}u))\Big)\phi\, d\vec{x} = 0. \tag{2.3}$$

The approximate solution $u_h = u_h(\vec{x},t)$ is defined as a piecewise polynomial in space and time, represented in the grid cell Q_i by

$$u_i(\vec{x},t) = \sum_{l=1}^{\mathcal{N}}\hat{u}_{i,l}(t)\phi_{i,l}(\vec{x}), \tag{2.4}$$

where $\phi_{i,l} = \phi_{i,l}(\vec{x})$, $l = 1,\dots,\mathcal{N}$ are basis functions, which span the space of polynomials of degree N with support Q_i and $\hat{u}_{i,l}(t)$, $l = 1,\dots,\mathcal{N}$ are the time-dependent degrees of freedom. We use a set of orthonormal basis functions which are constructed using the Gram–Schmidt orthogonalization algorithm. The number of the degrees of freedom is independent of the shape of the grid cell and given by

$$\mathcal{N} = \frac{1}{d!}\prod_{j=1}^{d}(N+j), \tag{2.5}$$

with space dimension d and polynomial degree N, see [16] for details.

The integration by parts with respect to the space variables is performed yielding the weak form of the DG scheme

$$\int_{Q_i}(u_i)_t\phi\, d\vec{x} - \int_{Q_i}\vec{f}^a\cdot\vec{\nabla}\phi\, d\vec{x} + \int_{Q_i}\mu\vec{\nabla}u_i\cdot\vec{\nabla}\phi\, d\vec{x} + \int_{\partial Q_i}\Big(\vec{g}^a - \vec{g}^d\Big)\cdot\vec{n}\phi\, ds + \int_{\partial Q_i} g^s\Big[\vec{\nabla}\phi\cdot\vec{n}\Big]^- ds = 0, \tag{2.6}$$

where $\vec{n}$ denotes the outward pointing normal vector of the element faces ∂Q_i and $[\cdot]^-$ refers to an evaluation at the boundary from the interior of the grid cell.

As the solution is discontinuous across the grid cell faces, numerical flux functions $\vec{g}$ are introduced in the surface integral and represent the coupling between the grid cells. Here, $\vec{g^a}$ denotes the numerical advection flux. In all our calculations we used a Godunov-type flux, the Roe flux or the HLLC flux, which are described in the book of Toro [41] in detail. There are also several suggestions for a proper diffusion flux proposed in the literature. The first one for the Navier–Stokes equations was proposed by Bassi and Rebay [5] and was generalized by Cockburn and Shu [10], named local discontinuous Galerkin method. Another approach was studied for elliptic equations, called the interior penalty method and described in a unified formulation in [2]. A numerical diffusion flux, which is well justified within the finite volume approach, was proposed in [18,30]. This diffusion flux $\vec{g^d}$ is computed from the solution of the diffusive Generalized Riemann Problem (dGRP) and is based on an approximate Riemann solution for diffusion. This flux has an additional scalar term $g^s := \mu u_i - [\mu u_i]^-$, involving the jump of the functional values. It is similar to the symmetric interior penalty scheme and is adjoint-consistent with optimal order of convergence, see [19] for details with respect to the Navier–Stokes equations. The boundary conditions are weakly imposed by specifying the corresponding numerical fluxes.

We choose the test functions from the full set of basis functions and obtain for each grid cell Q_i a system of ordinary differential equations (ODE) in time for the $\mathcal{N}$ degrees of freedom:

$$(\underline{\hat{u}}_i)_t(t) = \mathcal{R}_S\big(\underline{\hat{u}}_i(t),\underline{\hat{u}}_i^+(t)\big) + \mathcal{R}_V(\underline{\hat{u}}_i(t)). \tag{2.7}$$

Here, $\underline{\hat{u}}_i(t)$ denotes the degrees of freedom written as a vector, $\mathcal{R}_V(\underline{\hat{u}}_i)$ comprises the volume integrals and $\mathcal{R}_S(\underline{\hat{u}}_i,\underline{\hat{u}}_i^+)$ represents the surface integrals, which also depend on neighbor data $\underline{\hat{u}}_i^+$. Considering the global approximate solution, we get a weakly coupled system of ordinary differential equations for the time dependent degrees of freedom.

2.2. Space–time formulation

To complete the final DG scheme, the semi-discrete formulation (2.7) has to be integrated in time as well. In principle, any numerical method to solve a system of ordinary differential equations can be applied. For an explicit time approximation a well-known approach is the use of a Runge Kutta method, proposed by