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An anisotropic adaptive finite element algorithm for transonic viscous flows around a wing

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ABSTRACT

An adaptive finite element algorithm to compute transonic viscous flows around a wing is presented. The adaptive criteria is based on an anisotropic error estimator in the H^1 semi-norm, justified for an advection–diffusion problem with stabilized finite elements. The mesh aspect ratio can be arbitrarily large, upper and lower bounds can be proved, the involved constants being aspect ratio independent.

Based on this error estimator, an anisotropic mesh adaptation algorithm is proposed to compute transonic viscous flows around a wing. The mesh is structured around the wing, while the remaining part of the mesh is adapted according to the anisotropic error estimator. This anisotropic adaptive algorithm allows shocks and the wake to be captured accurately, while keeping the number of vertices as low as possible.

1. Introduction

Adaptivity with strongly anisotropic meshes is particularly suited to solving partial differential equations with internal or boundary layers. The use of strongly anisotropic meshes involves several theoretical and practical issues: design of stable discretization methods [1], error estimates involving constants which do not depend on the mesh aspect ratio [2–6], convergence of solvers [7], generation of anisotropic meshes [8–11]. The possible gain in memory and computing time is so important that previously intractable problems may become tractable [12].

The set of partial differential equations corresponding to transonic viscous flows is therefore a good candidate to be solved with anisotropic meshes, due to the presence of both boundary layers and shocks. Strongly anisotropic structured meshes are already widely used in the aeronautic industry for the numerical simulation of transonic viscous bodies [13–15]. However, the computation of transonic flows on anisotropic structured meshes around the body and anisotropic unstructured adaptive meshes elsewhere, see Fig. 1, has only been tackled recently [16,17]; it is precisely the goal of this paper.

The outline of the paper is the following. An anisotropic error estimator is presented for a diffusion-convection model problem in the next Section. Upper and lower bounds can be proved, the involved constants being aspect ratio independent. In Section 3, the viscous transonic flow around the ONERA M6 wing, at Reynolds number $11.72e6$, Mach number 0.84 and incident angle 3.06° is described. Numerical results are presented on industrial non-adapted anisotropic meshes provided by Dassault Aviation. Finally, an adaptive algorithm based on the anisotropic error estimator of Section 2 is proposed. The mesh is kept structured and anisotropic around the wing, while the remaining part of the mesh is adapted according to the anisotropic error estimator. Numerical results with such anisotropic adapted meshes are then presented.

2. A model problem: advection–diffusion

An advection–diffusion model problem is introduced in order to justify the use of our adaptive criterion. For the sake of clarity, this model problem is presented in two space dimensions; it can easily be extended in three space dimensions, all the numerical experiments in this paper being in three space dimensions.

Let Ω be a polygon of $\mathbb{R}^2$, with boundary $\partial\Omega$, let $\epsilon > 0$ be the diffusion coefficient, $\mathbf{a} = (a_1, a_2)^T \in \mathbb{R}^2$ the constant velocity field, $f \in L^2(\Omega)$ the source term. We are searching for $u : \Omega \to \mathbb{R}$ such that:

$$\begin{cases} -\epsilon\, \Delta u + \mathbf{a} \cdot \nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \nabla u \cdot \mathbf{n} = 0 & \text{on } \Gamma_2. \end{cases} \tag{1}$$

Here we have set $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap \Gamma_2 = \emptyset, \Gamma_1$ has non zero measure and $\mathbf{n}$ is the unit outer normal. Assuming $\mathbf{a} \cdot \mathbf{n} = 0$ on Γ_2, we have

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$$\int_{\Omega} (\mathbf{a} \cdot \nabla u) u = 0,$$

thus the Lions–Lax–Milgram theorem applies and the above problem has a unique weak solution $u \in V = \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma_1\}$. For all $h > 0$, consider $\mathcal{T}_h$ a conformal mesh of $\overline{\Omega}$ into triangles K with diameter h_K less than h. In this paper, anisotropic triangles are used, that is to say triangles with possibly large aspect ratio. The framework of [3,4] is used to describe the mesh anisotropy, although the one of [2] could also be used. Let $T_K : \hat{K} \to K$ be the affine mapping from the reference triangle $\hat{K}$ to an arbitrary triangle K. For any $\hat{\mathbf{x}} \in \hat{K}$, let $\mathbf{x} \in K$ be the corresponding vector in triangle K defined by

$$\mathbf{x} = T_K(\hat{\mathbf{x}}) = M_K \hat{\mathbf{x}} + \mathbf{t}_K, \tag{2}$$

where $\mathbf{t}_K \in \mathbb{R}^2$ and M_K is the jacobian matrix. Since M_K is invertible, it admits a singular value decomposition

$$M_K = R_K^T \Lambda_K P_K, \tag{3}$$

where R_K and P_K are orthogonal matrices and Λ_K is a diagonal matrix with positive entries:

$$\Lambda_K = \begin{pmatrix} \lambda_{1,K} & 0 \\ 0 & \lambda_{2,K} \end{pmatrix} \quad \text{and} \quad R_K = \begin{pmatrix} \mathbf{r}_{1,K}^T \\ \mathbf{r}_{2,K}^T \end{pmatrix}, \tag{4}$$

having chosen $\lambda_{1,K} \geqslant \lambda_{2,K}$. The unit vectors $\mathbf{r}_{1,K}$ and $\mathbf{r}_{2,K}$ correspond to the directions of maximum and minimum stretching, the scalars $\lambda_{1,K}$ and $\lambda_{2,K}$ correspond to the amplitudes of maximum and minimum stretching. In the framework of anisotropic meshes, the ratio between $\lambda_{1,K}$ and $\lambda_{2,K}$ can be large, however, the number of neighbours of a given vertex of the mesh should be bounded above. Moreover, due to the use of Clément interpolant, there is a technical condition involving Δ_K, the union of triangles sharing a vertex with K, see [18] for instance; this condition is satisfied whenever the variations in the direction of stretching are smooth throughout the mesh.

Let V_h be the finite element subspace of V corresponding to continuous, piecewise linear functions on the triangles of $\mathcal{T}_h$. The following stabilized finite element formulation is considered: find $u_h \in V_h$ such that

$$\int_{\Omega} \epsilon \nabla u_h \cdot \nabla v_h + \int_{\Omega} \mathbf{a} \cdot \nabla u_h \, v_h + \sum_{K \in \mathcal{T}_h} \tau_K \int_K (\mathbf{a} \cdot \nabla u_h)(\mathbf{a} \cdot \nabla v_h)$$
$$= \int_{\Omega} f \, v_h + \sum_{K \in \mathcal{T}_h} \tau_K \int_K f(\mathbf{a} \cdot \nabla v_h), \quad \forall v_h \in V_h. \tag{5}$$

The definition of the stabilization parameter τ_K has to be updated in order to comply with mesh anisotropy. Rather than the usual definition for isotropic meshes [19], the theoretical and numerical study performed in [1] have shown that the following choice yields accurate results:

$$\tau_K = \frac{\lambda_{2,K}}{2|\mathbf{a}|_\infty} \min(1, Pe_K) \tag{6}$$

with $|\mathbf{a}|_\infty = \max(|a_1|, |a_2|)$, Pe_K the Péclet number:

$$Pe_K = \frac{|\mathbf{a}|_\infty \lambda_{2,K}}{6\epsilon}. \tag{7}$$

In order to define our error estimator, we need some more notations. The L^2 projection onto the piecewise constants needs to be introduced; for any $K \in \mathcal{T}_h$, let

$$\Pi_K f = \frac{1}{|K|} \int_K f, \tag{8}$$

and let $|\ell_{i,K}|, i = 1, 2, 3$ be the three lengths of edges of K. Then, the anisotropic error estimator in the H^1 semi-norm is defined on triangle K by

$$\eta_K^2 = \left(\frac{1}{\epsilon} \|\Pi_K f - \mathbf{a} \cdot \nabla u_h\|_{L^2(K)} + \frac{1}{2} \sum_{i=1}^{3} \left(\frac{|\ell_{i,K}|}{\lambda_{1,K}\lambda_{2,K}} \right)^{1/2} \|[\nabla u_h \cdot \mathbf{n}]\|_{L^2(\ell_{i,K})} \right)$$
$$\times \omega_K(u - u_h), \tag{9}$$

where $[\cdot]$ denotes the jump of the inside value across an internal edge, $[\cdot] = 0$ if the edge belongs to Γ_1, $[\cdot]$ equals twice the inside

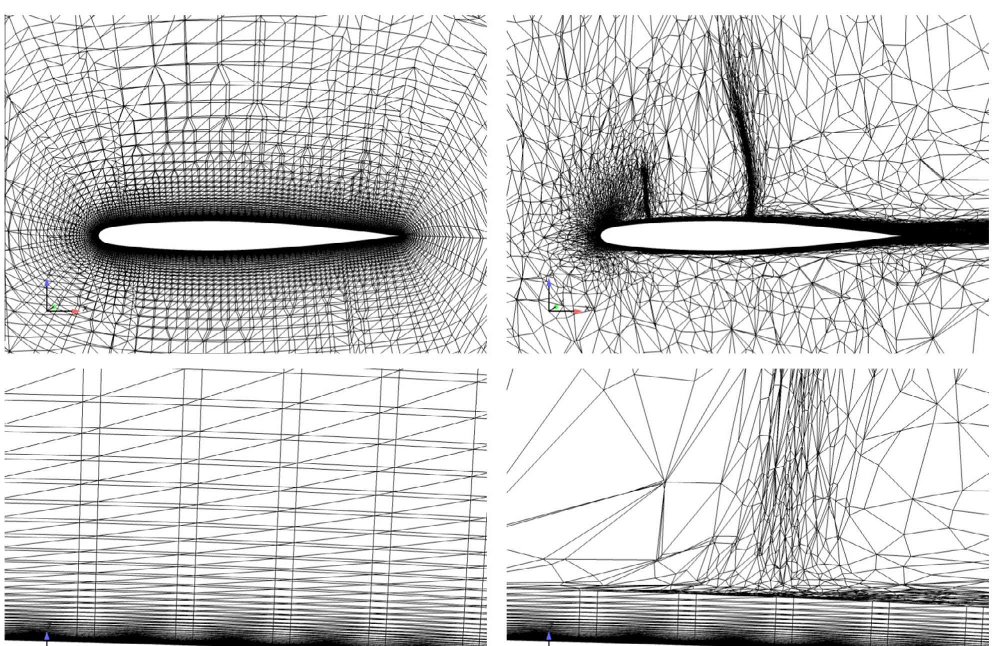

Fig. 1. Examples of anisotropic meshes for transonic viscous flows around a wing. Left: structured mesh, right: structured mesh around the wing, adapted mesh elsewhere. Bottom: close view around the wing, at the intersection between the shock and the boundary layer.