Computers & Fluids 105 (2014) 1–7

Contents lists available at ScienceDirect

Computers & Fluids

journal homepage: www.elsevier.com/locate/compfluid

Mesh dependent stability of discretization of the streamer equations for very high electric fields

Andrea Villa [a,*], Luca Barbieri [a], Marco Gondola [a], Andres R. Leon-Garzon [b], Roberto Malgesini [a]

[a] Ricerca Sul Sistema Energetico (RSE), Via Rubattino 50, 20134 Milano, Italy
[b] CMIC Department "Giulio Natta", Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy

ARTICLE INFO

Article history:
Received 24 February 2014
Received in revised form 21 July 2014
Accepted 2 September 2014
Available online 16 September 2014

Keywords:
Advection
Reaction
Streamer

ABSTRACT

The simulation of streamer evolution has many technological applications ranging from pollution reduction to surface treatments. In this paper we study an instability that may affect the discretized form of the streamer model when a very high electric field is considered and when a representative finite volume method is used. It is shown that the instability is caused by the advection–reaction terms that represent the transport and the production of electrons. It is proved that the mesh spacing greatly affects the properties of the discrete method in terms of stability and accuracy. A stabilization technique is introduced and tested. The proposed method is effective even when coarse meshes are used.

1. Introduction

In this paper we analyze some aspects of the stability of a finite volume scheme for the discretization of the streamer model, see [1], when very high electric fields are considered. To be more precise, we study the stability of the discretization of a linear advection–reaction equation that represents the transport of the electrons and the impact ionization. The peculiarity of this equation is that the reaction term is very high and positive thus leading to exponentially growing solutions in space.

The numerical simulation of streamers has many technical applications such as, for instance, surface treatments [2,3], prediction of the evolution of partial discharges [4,5] and pollution reduction [6].

In most of the cases found in the literature the streamer problem has been tackled using two types of solvers: a Poisson solver and a solver for a set of advection–reaction equations. Many approaches have been proposed for the latter problem. One of the first implemented for this particular problem was the finite difference method [7]. Another very popular method is the flux corrected transport one, see [8] for a general description of the method and [9–11] for some applications. In recent years some stabilized versions of the finite element method have been used too [12,13]. Also high resolution techniques have been applied to the simulation of discharge inception: for instance the discontinuous Galerkin method has been used successfully in [14].

Whatever the chosen simulation technique, in all these cases a very fine grid is used with a characteristic mesh size of nearly one micro-meter. In some cases also the adaptive mesh refinement technique is used to guarantee a sufficient resolution near the head of the streamer, see [15–17]. From an heuristic point of view the mesh spacing used in the above mentioned works ensures the stability of the numerical schemes, however a proper analysis is still lacking.

To this end, in this work we study the stability of a reference finite volume scheme with respect to mesh spacing. Our choice to analyze a finite volume scheme has been dictated by the fact that a large part of the numerical solvers for the streamer simulation are based on that method. Moreover the finite volume schemes can manage easily the strong gradients of the ionization fronts. Finally the finite volume schemes fit well in the framework of the asymptotic preserving schemes as has been demonstrated in [18,19] for the Euler equations and, specifically, in [20] for the streamer equations. The asymptotic preserving schemes guarantee good stability performances even when the time integration step is much larger than the plasma relaxation time.

We prove that the discretization of a linear advection–reaction equation, with a positive advection coefficient, may lead to unstable finite volume schemes and we analyze how to stabilize them. For completeness we also state that the characteristic methods [21] and some of their evolutions such as the ELLAM technique [22] can provide conservative and stable schemes even when the reaction coefficient is positive. However those schemes cannot

* Corresponding author.
E-mail addresses: andrea.villa@rse-web.it (A. Villa), luca.barbieri@rse-web.it (L. Barbieri), marco.gondola@rse-web.it (M. Gondola), andresricardo.leon@polimi.it (A.R. Leon-Garzon), roberto.malgesini@rse-web.it (R. Malgesini).

http://dx.doi.org/10.1016/j.compfluid.2014.09.002
0045-7930/

be easily incorporated in the asymptotic preserving methods and, if the discontinuity fronts are not explicitly traced, they are, in general, more sensitive to steep gradients.

In the literature there is a huge amount of works dealing with the analysis of the stability of diffusion–advection–reaction equations, see, among the many paper, [23–25]. On the contrary, less works are devoted to the advection–reaction problem, see [26]. Moreover, to the best of our knowledge, there is no work devoted to the analysis of the stability and the accuracy of the discretization of streamer solvers when a very high field is applied.

In this work we introduce a simplified form of the streamer model for high electric fields. We consider, for the sake of simplicity, a one dimensional model. We concentrate ourselves on the drift-reaction equation with a fixed electric field (the coupling between the electric field solver and the drift equation has been already treated in [18–20]). We show that the mesh size plays a crucial role in the stability of finite volume schemes and we propose a stabilization technique that makes them bounded, whatever the mesh size. In the second part of the paper we study the numerical properties of the scheme we have implemented. Moreover, since we are aiming to expand our method to 3D cases, we also propose a technique to enhance its accuracy without resorting to a mesh refinement. In fact, this would imply an unacceptable numerical burden in three dimensions.

Let us review the structure of this paper. In Section 2 we introduce the continuous streamer problem and we isolate the drift-reaction part. The properties of the continuous model are studied. In Sections 3 and 4 we introduce the discretized form and we study its properties. In Section 5 we introduce some numerical experiments to back the theoretical estimations and we compare our approach with an implementation of the characteristic method. In Section 6, we critically review the results obtained.

2. The continuous problem

Let us introduce a simplified version of the streamer model [20,1] when a very high electric field is considered:

$$\begin{cases} \frac{\partial n_e}{\partial t} - \frac{\partial}{\partial x}\left(n_e \mu_e E\right) = \alpha \mu_e |E| n_e, \\ \frac{dn_p}{dx} = \alpha \mu_e |E| n_e, \\ \frac{dE}{dx} = \frac{e}{\varepsilon_0}\left(n_p - n_e\right), \\ -\frac{d\phi}{dx} = E, \end{cases} \tag{1}$$

where $n_e(t,x)$, $n_p(t,x)$ are the concentrations of electrons and positive ions respectively, $t \in (0, t^{fin}]$ is the time, t^{fin} is the final time (not necessarily bounded), $x \in [0, L]$ is the spatial variable, L is the length of the domain, $\mu_e > 0$ is the mobility of electrons, $\alpha > 0$ is a reaction coefficient representing the impact ionization, $E(t,x)$ is the electric field, $\phi(t,x)$ is the electric potential, e is the electron charge and, finally, ε_0 is the vacuum permeability.

If we compare model (1) with the one proposed in [20,1] we have neglected the negative ions, the recombination terms and the electron attachment.

System (1) is completed by a set of boundary conditions, in particular we impose:

$$\begin{cases} n_e(t,0) = n_{e,b}(t), & \text{if } E(t,0) < 0, \\ n_e(t,L) = n_{e,b}(t), & \text{if } E(t,L) > 0, \\ \phi(t,0) = \phi_b(t), \\ E(t,L) = E_b(t), \end{cases} \tag{2}$$

where $n_{e,b}$ is a boundary concentration value and ϕ_b and E_b are the boundary potential and electric field respectively. As regards the initial conditions we impose $n_e(0,x) = n_e^0(x)$ and $n_p(0,x) = n_p^0(x)$, where n_e^0 and n_p^0 are bounded positive functions.

As we have already said, we concentrate our attention on the first equation of (1) and we simplify the notation to get

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \alpha c u, \tag{3}$$

where u is the solution and $c > 0$ is a constant advection speed. In other words, Eq. (3) is a constant coefficient linear equation which has to be complemented with an inflow boundary condition $u(t,0) = u_b$, where, for the sake of simplicity, u_b is a time-constant boundary condition, and $u_0(x)$ is an initial condition such that $u(0,x) = u_0(x)$.

The solution of Eq. (3) is bounded as long as $L < \infty$. In fact its exact solution is

$$\begin{cases} u(t,x) = u_0(x - ct)e^{\alpha c t}, & \text{if } x - ct > 0, \\ u(t,x) = u_b e^{\alpha x}, & \text{otherwise.} \end{cases} \tag{4}$$

Therefore we have

$$\|u(t)\|_{L^\infty(0,L)} \leqslant \max\left(u_b e^{\alpha L}, \|u_0\|_{L^\infty(0,L)} e^{\alpha L}\right), \quad t \in (0, t^{fin}]. \tag{5}$$

3. The discretization

3.1. Stability condition

We now introduce a discretization for Eq. (3). Let us consider a uniform grid with N_c cells τ_k, $k = 1, \ldots, N_c$, with cells of size $h = L/N_c$ and a uniform subdivision $t^0, \ldots, t^n$ (with n, in principle, unbounded) of the time interval $(0, t^{fin}]$ with constant time steps of size Δt. We define U_k^n the mean value of the discrete solution on the cell τ_k at time step n where U_0^k with $k = 1, \ldots, N_c$ is an approximation of the initial condition. Let

$$v = \frac{c\Delta t}{h}, \tag{6}$$

be the Courant number: we choose Δt such that $v < 1$ and the scheme satisfies the Courant–Friedrichs–Lewy (cfl) condition, see [27].

We consider a first order finite volume discretization of (3). Therefore we integrate (3) in space on a generic cell τ_k in the interval $[t^n, t^{n+1}]$ and we get

$$h\left(U_k^{n+1} - U_k^n\right) + \int_{t^n}^{t^{n+1}} cu(t, hk) - cu(t, h(k-1)) = \int_{t^n}^{t^{n+1}} \int_{\tau_k} \alpha c u. \tag{7}$$

Then we approximate the interface fluxes:

$$\int_{t^n}^{t^{n+1}} cu(t, hk) \approx \Delta t c U_k^n, \quad \int_{t^n}^{t^{n+1}} cu(t, h(k-1)) \approx \Delta t c U_{k-1}^n, \tag{8}$$

and we approximate the source term $\int_{t^n}^{t^{n+1}} \int_{\tau_k} \alpha c u \approx \Delta t h \alpha c U_k^n$. Then we obtain

$$U_k^{n+1} = (1 - v)U_k^n + vU_{k-1}^n + \alpha c \Delta t U_k^n. \tag{9}$$

We can show that scheme (9) is unstable. In fact, supposing that the discrete solution is positive, then

$$U_k^{n+1} = (1 - v + \alpha c \Delta t)U_k^n + vU_{k-1}^n \geqslant (1 - v + \alpha c \Delta t)U_k^n. \tag{10}$$

Therefore if

$$1 - v + \alpha c \Delta t > 1, \tag{11}$$

the solution of (9) in a generic kth cell is unbounded. Using (6), it can be verified that condition (11) is equivalent to

$$h\alpha > 1. \tag{12}$$

This means that, using too coarse meshes, method (9) becames unstable. To the best of our knowledge, this mesh dependent