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Comparison between homotopy perturbation method and optimal homotopy asymptotic method for the soliton solutions of Boussinesq–Burger equations

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ABSTRACT

In this article, a comparative study between homotopy perturbation method (HPM) and optimal homotopy asymptotic method (OHAM) is presented. Homotopy perturbation method is applied to compute the numerical solutions of non-linear partial differential equations like Boussinesq–Burger equations. The approximate solutions of the Boussinesq–Burger equation are compared with the optimal homotopy asymptotic method as well as with the exact solutions. Comparison between our solutions and the exact solution shows that both the methods are effective and accurate in solving nonlinear problems whereas OHAM is accurate with less number of iterations in compared to HPM. In OHAM the convergence region can be easily adjusted and controlled. OHAM provides a simple and easy way to control and adjust the convergence region for strong nonlinearity and is applicable to highly nonlinear fluid problem like Boussinesq–Burger equations.

1. Introduction

With the development of science and engineering, nonlinear evolution equations have been used as the models to describe physical phenomena in fluid mechanics, plasma waves, solid state physics, chemical physics, etc. So, for the last few decades, a great deal of attention has been directed towards the solution (both exact and numerical) of these problems. Various methods are available in the literature for the exact and numerical solution of these problems.

Perturbation methods have been used for the solution of nonlinear problems in science and engineering [1–4]. But, most of the perturbation techniques require the existence of a small parameter in the equation. An unsuitable choice of such parameter would lead to very bad results. The solutions obtained through perturbation methods can be valid only when a small value of the parameter is used. Hence, it is necessary to check validity of the approximations through numerical processes.

In HPM and OHAM, the concept of homotopy from topology and conventional perturbation technique were merged to propose a general analytic procedure for the solution of nonlinear problems. Thus, these methods are independent of the existence of a small parameter in the problem at hand and thereby overcome the limitations of conventional perturbation technique. OHAM, however, is the most generalized form of HPM as it employs a more general auxiliary function $H(p)$ in place of HPM's p.

The OHAM was introduced and developed by Marinca et al. [5–7] and it can be shown that HPM is a special case of OHAM. Several authors have proved the effectiveness, generalization and reliability of this method. The advantage of OHAM is built in convergence criteria, which is controllable. In OHAM, the control and adjustment of the convergence region are provided in a convenient way.

In the proposed OHAM procedure, the construction of the homotopy is quite different. In the frame of OHAM the linear operator L is well defined by Eq. (4.5) and the initial approximation is determined rigorously. Instead of an infinite series, the OHAM searches for only a few terms (mostly two or three terms). The way to ensure the convergence in OHAM is quite different and more rigorous. Unlike other homotopy procedures, OHAM ensure a very rapid convergence since it needs only two or three terms for achieving an accurate solution. This is in fact the true power of the method. The auxiliary constants $C_1, C_2, C_3, \ldots$ provide us a convenient way to guarantee the convergence of OHAM series solution. Moreover, the optimal values of convergence control constants guarantee the certain convergence of OHAM series solution.

Generalized Boussinesq–Burger equation [8–10] is a nonlinear partial differential equation of the form

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$$u_t - \frac{1}{2}v_x + 2uu_x = 0, \tag{1.1}$$

$$v_t - \frac{1}{2}u_{xxx} + 2(uv)_x = 0, \quad 0 \leqslant x \leqslant 1 \tag{1.2}$$

The Boussinesq–Burger equations arise in the study of fluid flow and describe the propagation of shallow water waves. Here x and t respectively represent the normalized space and time, $u(x,t)$ is the horizontal velocity field and $v(x,t)$ denotes the height of the water surface above a horizontal level at the bottom.

Various analytical methods such as Darboux transformation method [11], Lax pair, Bäcklund transformation method [12] have been used in attempting to solve Boussinesq–Burger equations. Our aim in the present work is to implement homotopy perturbation method (HPM) and optimal homotopy asymptotic method (OHAM) in order to demonstrate the capability of these methods in handling system of nonlinear equations, so that one can apply it to various types of nonlinearity.

This paper is systematized as follows: in Section 1, introduction to Boussinesq–Burger equation is described. In Sections 2 and 3, the mathematical preliminaries of homotopy perturbation method (HPM) and its convergence are presented. The basic idea of optimal homotopy asymptotic method (OHAM) and its convergence is discussed in Sections 4 and 5 respectively. Next we applied the HPM and OHAM for solving Boussinesq–Burger equations in Sections 6 and 7 respectively. The numerical results and discussions are presented in Section 8 and Section 9 concludes the paper.

2. Basic idea of homotopy perturbation method (HPM) and He's polynomials

A combination of the perturbation method and the homotopy method is called the homotopy perturbation method (HPM), which has eliminated the limitations of traditional perturbation methods.

To illustrate the basic ideas of homotopy perturbation method [13] we consider the following nonlinear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{2.1}$$

with the boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma \tag{2.2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is known as analytic function, and Γ is the boundary of the domain Ω.

The operator A can be divided into two parts linear L and nonlinear N. Therefore Eq. (2.1) can be rewritten as follows

$$L(u) + N(u) - f(r) = 0 \tag{2.3}$$

We construct a homotopy $v(r,p)$ of Eq. (2.1) as follows $v(r,p) : \Omega \times [0,1] \to \Re$ which satisfies [14]

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0,$$
$$p \in [0,\ 1], \quad r \in \Omega \tag{2.4}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{2.5}$$

where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of Eq. (2.1), which satisfies the boundary conditions. It follows from (2.4) and (2.5) that

$$H(v,0) = L(v) - L(u_0) = 0, \tag{2.6}$$

$$H(v,1) = A(v) - f(r) = 0 \tag{2.7}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(v) - L(u_0)$, $A(v) - f(r)$ are called homotopic.

We assume that the solution of Eq. (2.5) can be written as a power series in p

$$v = v_0 + pv_1 + p^2v_2 + \cdots \tag{2.8}$$

The approximate solution of Eq. (2.1) can be obtained by setting $p = 1$

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots \tag{2.9}$$

The nonlinear term $N(u)$ can be expressed in He polynomials [15] as

$$N(u) = \sum_{m=0}^{\infty} p^m H_m(v_0, v_1, \ldots, v_m) \tag{2.10}$$

where

$$H_m(v_0, v_1, \ldots, v_m) = \frac{1}{m!}\frac{\partial^m}{\partial p^m}\left(N\left(\sum_{k=0}^{m} p^k v_k\right)\right)\Bigg|_{p=0}, \quad m = 0,1,2,\ldots \tag{2.11}$$

The series (2.9) is convergent for most cases. The following suggestions has been made by He [16] to find the convergence rate on nonlinear operator $A(v)$.

(i) The second derivative of $N(u)$ with respect to u must be small because the parameter may be relatively large, i.e. $p \to 1$.
(ii) The norm of $L^{-1}\frac{\partial N}{\partial u}$ must be smaller than one so that the series converges.

3. Convergence of HPM [16]

Let us write Eq. (2.5) in the following form

$$L(v) = L(u_0) + p[f(r) - N(v) - L(u_0)] \tag{3.1}$$

Applying the inverse operator, L^{-1} to both sides of Eq. (3.1), we obtain

$$v = u_0 + p[L^{-1}f(r) - L^{-1}N(v) - u_0] \tag{3.2}$$

Suppose that

$$v = \sum_{i=0}^{\infty} p^i v_i \tag{3.3}$$

Substituting (3.3) into the right-hand side of Eq. (3.2), we have

$$v = u_0 + p\left[L^{-1}f(r) - (L^{-1}N)\left[\sum_{i=0}^{\infty} p^i v_i\right] - u_0\right] \tag{3.4}$$

If $p \to 1$, the exact solution may be obtained by using

$$u = \lim_{p\to 1} v = L^{-1}f(r) - (L^{-1}N)\left[\sum_{i=0}^{\infty} v_i\right]$$
$$= L^{-1}f(r) - \sum_{i=0}^{\infty}(L^{-1}N)v_i \tag{3.5}$$

To study the convergence of the method let us state the following Theorem.

Theorem 1. *Suppose that V and W be Banach spaces and $N:V \to W$ be a contraction mapping such that for all*

$$v, v^* \in V; \|N(v) - N(v^*)\| \leqslant \varepsilon\|v - v^*\|, \quad \varepsilon \in (0,1) \tag{3.6}$$

Then according to Banach's Fixed point theorem N has a unique fixed point u (say) such that $N(u) = u$.

The sequence generated by the homotopy perturbation method will be assumed in the following form as