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Conservative time integrators of arbitrary order for skew-symmetric finite-difference discretizations of compressible flow

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ABSTRACT

Skew-symmetric discretizations of the Navier–Stokes equations avoid the introduction of artificial numerical damping by first principles and are thus attractive for the simulation of turbulence and flow induced sound. For direct numerical simulations a high discretization order in space and time is essential to obtain high quality results at affordable cost. While skew-symmetric and fully conservative schemes of high spatial discretization order are available, the time integrators respecting skew-symmetry and conservation are of second order only. We present a class of time integrators for skew-symmetrical schemes for compressible flow, which allow an arbitrary order of accuracy. We also show that these schemes are discretely norm-conserving. Test cases for isotropic turbulence are performed. A comparison with standard spatial and temporal discretization completes our survey.

1. Introduction

When treating the evolution equations for fluid flow numerically, all schemes try to emulate the analytical equations as well as possible. Apart form the basic properties of numerical schemes, like consistency and accuracy, the Navier–Stokes equations offer an array of structural properties that numerical schemes try to preserve. These are the conservation properties of mass, momentum and energy, which are the core of finite-volume methods derived from the integral formulation of the equations, the dispersion and wave-propagation relations whose correct treatment are the hallmarks of high-order finite and spectral differences. As is natural, different numerical ansatzes are suited to different physical problems. Conservative finite-volume methods are a great way to handle configurations containing shocks, e.g. supersonic jets or nozzles, but these schemes often employ upwinding for stability reasons. Thus leading to high numerical damping which makes it difficult to extract useful information about the propagation of sound from the simulations. On the other hand, high-order finite-difference schemes with very low or no numerical dissipation might be well suited for the simulation of acoustics but will fail in the vicinity of shocks. This makes it especially difficult when treating situations where physical phenomena with competing numerical demands occur, such as shock-wave/boundary-layer interactions or simulations of supersonic jet noise. One approach to this perceived dilemma is the so-called *skew-symmetric* or *split-convective* form of the equations where the convection term in the momentum equation is split into a skew-symmetric operator to reduce aliasing errors and enhance stability properties of the discretization, see [1–3]. The classical ansatz of skew-symmetric splitting in fluid dynamics, introduced by Morinishi et al. [4], uses cleverly averaged quantities to ensure the *telescoping sum property* of the skew-symmetric operator and through this achieves conservation by first principles. This ansatz has been applied both to the compressible, e.g. [5,6], and incompressible equations, e.g. [4,7]. These ideas have also been employed in the finite-volume context, e.g. [8–10], and recently also using Discontinuous-Galerkin discretizations in [11]. Splitting the convection operator can be shown to improve energy conservation, especially in the inviscid limit and casts the operators in locally conservative form, [1]. In [12], Reiss and Sesterhenn developed a spatial discretization of the skew-symmetric Navier–Stokes equations that is matrix-based and achieves conservation without additional averaging procedures. The eschewal of special averages in the discretization opens up the use of a wide range of spatial discretization schemes and introduces the possibility of adding split-convective operators into existing finite-difference codes. The extension of these schemes to distorted or moving grids in the context of finite-differences is not straight-forward. On the basis of ideas introduced by Kok in [13,14] both Morinishi et al. and Reiss were

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able to extend their respective formulations to distorted grids in [15,12]. This extension marks another significant step for skew-symmetric, conservative discretizations towards applicability for practical problems. While these approaches are very successful when dealing with the discretization in space, retaining the perfect conservation properties in a discrete time setting has been difficult, being restricted to implicit schemes of 2nd order or multi-step methods which only conserve time-averages of the desired quantities and thus are not unconditionally stable. Within this paper we will present a formulation of the skew-symmetric Navier–Stokes equation that is fully conservative in space and time, using the simple matrix-type discretization developed in [12] as a baseline for the spatial discretization and generalizing the idea of a *density-weighted velocity* introduced in Morinishi's and Subbareddy's work, [16,10], to achieve conservation in time with schemes of arbitrary order. Putting together these building blocks leads to a numerical scheme that combines a number of favorable traits. The use of high order central differences for the spatial discretization introduces no artificial dissipation making the simulation of acoustical phenomena feasible while the conservative nature of the scheme allows the correct treatment of shocks when suitable conservative filtering algorithms are applied to combat Gibbs oscillations. In addition, the conservation of total energy coupled with the conservation of mass is essentially equivalent to the conservation of a discrete norm of the system, ensuring stability in the absence of boundaries. We first introduce the basic ideas governing the skew-symmetric formulation, introduce the conservative semi-discretization that was developed by Reiss in Section 2 before presenting time integrators of arbitrary order in Section 3. Finally, in Section 4 we present numerical results of isotropic turbulence to asses the practical applicability of the scheme.

2. Skew-symmetric formulation of the Navier–Stokes equation

This section is concerned with the basic ideas of the skew-symmetric formulation and discretization of the Navier–Stokes equations. We motivate why observing skew-symmetry within the discrete equations leads to energy conservation and present the matrix-driven semi-discretization, developed in [12], that forms the baseline for all further work in this paper. We note that, for the sake of clarity, throughout this paper we will employ the Navier–Stokes equations in one-dimensional form on uniform grids. Periodicity of the domain is assumed. All results of this paper extend in a straight forward manner to distorted grids in multiple dimensions. More details on grid transformations can be found in [12] and the full 3-dimensional equations on distorted grids are given in the appendix. Before continuing, a note on how the terms conservative and conservation are used within this text is in order. This work is concerned with the temporal preservation of global quantities in the flow field. When ϕ is a generic function of the flow field, the quantities $\int \phi dx$ and its discrete approximation, $(\Delta x)\sum_i \phi_i$ - in the one-dimensional, periodic case - and their temporal evolutions are studied. The notion of a conservative scheme often refers to a numerical scheme that complies with the Lax–Wendroff theorem, and as such has local and consistent fluxes and is suitable to compute unsteady solutions. As the Lax–Wendroff theorem is concerned only with the spatial discretization of fluxes we refer the interested reader to [1,17] where the conservation properties of split-convective operators are studied in this context. In addition, [12] shows that local and consistent fluxes are implied by the skew-symmetric discrete equations employed in this text.

We begin by looking at the connection between skew-symmetry and energy conservation. As a first step we derive the skew-symmetric momentum equation as a combination of the convective and divergence formulations of the equation of momentum. Here, u is the velocity, ρ the density, p the pressure and τ denotes the one-dimensional viscous stress tensor.

$$\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = \partial_x \tau \qquad (divergence) \tag{1}$$
$$\iff \rho\partial_t u + \rho u \partial_x u + \partial_x p = \partial_x \tau \qquad (convection) \tag{2}$$

Taking $\frac{1}{2}((1)+(2))$ one obtains the skew-symmetric form of the momentum equation,

$$\frac{1}{2}(\partial_t \rho \cdot + \rho\partial_t \cdot)u + \frac{1}{2}(\rho u \partial_x \cdot + \partial_x \rho u \cdot)u + \partial_x p = \partial_x \tau \tag{3}$$

where now both the temporal and spatial derivative operator $\mathcal{A} = (\phi\partial \cdot + \partial\phi \cdot)$ are skew-symmetric as can be seen when multiplying with a testfunction v and integrating the scalar product $\int_\Omega v\mathcal{A}w\,dx$ by parts:

$$\int_\Omega v(\phi\partial_x \cdot + \partial_x\phi\cdot)w\,dx = -\int_\Omega w(\partial_x\phi\cdot + \phi\partial_x\cdot)v\,dx \tag{4}$$

where we have omitted boundary terms. This directly implies $\int_\Omega u\mathcal{A}u\,dx = 0$ which brings us back to the equation of momentum and energy conservation. The equation of momentum (3) can be transformed into an equation for the time evolution of total kinetic energy $E_{kin} = \int_\Omega \frac{1}{2}\rho u^2\,dx$ when multiplied by u and then integrated over the whole domain:

$$\partial_t \int_\Omega \frac{1}{2}(\rho u^2)\,dx = \int_\Omega -\left(\frac{1}{2}u(\rho u\partial_x \cdot + \partial_x \rho u\cdot)u + u\partial_x p - u\partial_x\tau\right)dx \tag{5}$$

which immediately reduces to the friction and pressure terms due to the skew-symmetry of the operator $\mathcal{A}$ and the periodicity of our domain, leaving us with

$$\partial_t \int_\Omega \frac{1}{2}(\rho u^2)\,dx = \int_\Omega (-u\partial_x p + u\partial_x\tau)\,dx \tag{6}$$

as the rate of change in kinetic energy. Combining this with the energy equation written in the internal energy $e = \left(\frac{1}{1-\gamma}p\right)$, where γ is the adiabatic index, one obtains conservation of total energy. The equation of internal energy, assuming the ideal gas law, $p = \rho RT$, can be written as

$$\frac{1}{1-\gamma}\partial_t p + \frac{\gamma}{1-\gamma}\partial_x(up) - u\partial_x p = \partial_x(u\tau) - u\partial_x\tau - \partial_x\Phi \tag{7}$$

where T is the temperature, R is the universal gas constant and $\Phi = -\lambda\partial_x T$ is the heat conduction with coefficient λ. Integrating over space leads to

$$\frac{1}{1-\gamma}\partial_t \int_\Omega p\,dx = \int_\Omega \left(-\frac{\gamma}{1-\gamma}\partial_x(up) + u\partial_x p + \partial_x(u\tau) - u\partial_x\tau - \partial_x\Phi\right)dx \tag{8}$$
$$\iff \frac{1}{1-\gamma}\partial_t \int_\Omega p\,dx = \int_\Omega (u\partial_x p - u\partial_x\tau)\,dx \tag{9}$$

Showing that total energy, $E = \left(\frac{1}{1-\gamma}p + \frac{1}{2}(\rho u^2)\right)$ is conserved due to the skew-symmetry of the convective term in (3) and the coupling of momentum and energy:

$$\partial_t \int_\Omega \left(\frac{1}{1-\gamma}p + \frac{1}{2}(\rho u^2)\right)dx = \int_\Omega (u\partial_x p - u\partial_x\tau) + (-u\partial_x p + u\partial_x\tau)\,dx = 0 \tag{10}$$

In the analytical case, these traits hold true for all formulations of the equations, but as the product rule is generally not true in the discrete, we need to discretize the skew-symmetric equation of momentum and the energy equation in a consistent manner when we want to preserve the skew-symmetry and thus the conservation of energy in our discrete formulation. Now we will show how to obtain a semi-discretization of these equations that will retain