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Parametric analysis of steady bifurcations in 2D incompressible viscous flow with high order algorithm

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ABSTRACT

This work deals with the computation of steady bifurcation points in 2D incompressible Newtonian fluid flows. The problem is modeled with the Navier–Stokes equations with an evolving geometric parameter. The aim of the present study is to propose a reliable and efficient numerical method for parametric steady bifurcation calculations. The numerical algorithm is based on the coupling of a continuation method with a homotopy technique. The continuation method lies on the asymptotic numerical method with Padé approximants for an initial linear stability analysis with an initial geometric configuration. The homotopy technique completes the calculation with the computation of critical Reynolds numbers for different discrete values of the geometric parameter. Two classical numerical problems are approached. The first one is the flow in sudden expansion. The geometric parameter is the height of the expansion inlet. The second problem is the flow in a divergent/convergent channel. In this case, the geometric parameter is the length of the channel. Comparisons of results with those obtained from the literature are performed, showing the efficiency of the proposed algorithm. The aim of this study is to determine the critical Reynolds numbers of the flow using few computations for each geometric parameter.

1. Introduction

The problem of a 2D steady Newtonian flow through an expansion has been widely studied in order to understand the well-known phenomenon of loss of symmetry [1–3]. This phenomenon is referred to as the instability or bifurcation phenomenon. One of the most influential attribute initiating instability is the critical Reynolds number Re_c: the flow, initially symmetric, becomes asymmetric for a Reynolds number value higher than Re_c.

Hence, a lot of studies concerning flow in symmetric channel with a sudden expansion can be found, see for example the pioneer experimental works of Durst et al. [4], Cherdron et al. [5] or numerical results of Fearn et al. [6] or Shapira et al. [7]. Since these first works, numerous studies have been proposed to study the stability of 2D flows. For example, Alleborn [1], Battaglia [2], Drikakis [3], Mizushima [8] or more recently Allery [9], Huang [10] or Lanzerstorfer [11] have studied stability of flow in channel. In these stability studies, the numerical algorithm is applied for all values of the geometric parameter. For example, in [11] the stability of flow in plane sudden expansion is studied with the expansion height as the geometric parameter. The stability analysis is performed for each geometric configuration using an eigenvalue computation. In [9], the authors use a bifurcation indicator to detect loss of symmetry in sudden expansion. Hence, for each value of the expansion height, a computation of the bifurcation indicator is carried out. Unfortunately, it is worth noticing that an indicator obtained using the asymptotic numerical method with a finite element discretization can require a lot of computation depending on the problem size.

In a context of parametric stability analysis with a progressing geometric parameter, several authors have proposed algorithms to solve such a problem for linear [12,13] or nonlinear vibrations of plates [14] and for buckling analysis [15].

This work deals with similar problems within the context of fluid mechanics. This study proposes to determine the evolution of critical Reynolds numbers for which a steady bifurcation takes places when a geometric parameter progresses. This work focuses on the detection of steady bifurcations for different discrete values of geometric parameters. The main question is how to use efficiently the result obtained in a configuration (Ω_i) to determine the critical Reynolds number for configuration (Ω_{i+1}).

The proposed algorithm couples a perturbation method with a homotopy technique. The perturbation method makes it possible to perform an initial linear stability analysis. The perturbation is computed using the asymptotic numerical method [16], which is

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not specifically devoted to parametric studies. Performance is enhanced by expanding solutions using Padé approximants [17,18].

The literature on the homotopy theory is extensive. As He [19] mentioned it, the Homotopy method consists in continuously deforming a simple problem, which is easy to solve, into the difficult problem under study. Such a technique is based on the expansion of the dependent variables and provides serie solutions. The technique has been successfully applied to obtaining the solution of a variety of nonlinear problems involving ordinary and partial differential equations [20].

In this study, as a complement to the initial calculation, the homotopy technique is applied [21]. Thus, the perturbation parameter is considered as the homotopy parameter. Doing so slightly modifies the initial problem. Firstly, convective terms appear as source terms which are written in the right hand side of the equations. Secondly, the problem possess the same initial tangent operator obtained from the perturbation calculation. The last property offers the opportunity to compute critical Reynolds number for different geometric configurations using the same tangent operator. As a consequence, computational time could be saved.

This paper is divided into three parts. In the first part, the equations governing the flow of an incompressible Newtonian fluid are recalled. The second part is dedicated to the linear stability analysis for the computation of steady bifurcations. The perturbation method and the homotopy technique are presented. The way these techniques are used is also given. Finally, in the third part, two academic problems are solved: the problem of the flow in sudden expansion and the problem of the flow in a divergent/convergent channel. The aim of these examples is to test algorithm abilities to compute the critical Reynolds number when a geometric parameter evolves. The geometric parameters are the height of the inlet expansion and the length of the channel, for, respectively, the first and the second problems. These parameters allow the definition of the expansion ratio $E = H/h$ and the aspect ratio $A = L/(3h)$ for, respectively, the first and the second problem. Results are obtained for differents discrete values of E and A and compared to results obtained with the continuation method [16] and a classical Newton iterative scheme. Numerical results make it possible to discuss the efficiency and the accuracy of the proposed algorithm.

2. Governing equations

The equations governing the steady flow of an incompressible Newtonian fluid are given by:

$$\begin{cases} -\nu u_{i,jj} + u_j u_{i,j} + \frac{1}{\rho} p_{,i} = 0 & \text{in } \Omega \\ u_{i,i} = 0 & \text{in } \Omega \\ u = \lambda u_b & \text{on } \partial_u \Omega \end{cases} \tag{1}$$

where u_i ($i = 1, 2$) and p stand, respectively, for the velocity components and the pressure. Parameter λ is the intensity of velocity u_b imposed on boundary $\partial_u\Omega$. This parameter is linked to the Reynolds number, Re, by considering a reference length, d, and the kinetic viscosity of the fluid, ν:

$$\mathrm{Re} = \frac{\lambda u_b d}{\nu} \tag{2}$$

The weak formulation associated with the Navier–Stokes Eq. (1) can be expressed with operators according to the following expression:

$$\mathbf{L}(U) + \mathbf{Q}(U, U) = \lambda F \tag{3}$$

In Eq. (3), U designates the mixed unknown vector containing the velocity Cartesian components u_i ($i = 1, 2$) and pressure p. Symbols $\mathbf{L}(U)$ and $\mathbf{Q}(U,U)$ stand for linear and quadratic operators containing, respectively, diffusive and convective terms of the Navier–Stokes Eq. (1). The right-hand side vector F in Eq. (3) is equivalent to the imposed velocity u_b applied on boundary $\partial_u\Omega$. A more precise definition of these operators is provided in [22]. In this paper, it is proposed to determine the Reynolds number for which the steady Navier–Stokes solutions, denoted by U^s, is not unique: this indicates that a steady bifurcation point appears for this critical Reynolds number Re_c. Hence, the steady solution U^s verifies the following nonlinear problem:

$$\mathbf{L}(U^s) + \mathbf{Q}(U^s, U^s) = \lambda F \tag{4}$$

The stability analysis of the steady solution U^s is carried out by a perturbation technique which consists in introducing a small disturbance V in the unknown U:

$$U = U^s + V \tag{5}$$

By introducing Eq. (5) into Eq. (3), neglecting the nonlinear terms of V and defining the quadratic operator $\mathbf{Q}^*(a,b) = \mathbf{Q}(a,b) + \mathbf{Q}(b,a)$, it is obtained the stability linearized equation:

$$\mathbf{L}(V) + \mathbf{Q}^*(U^s, V) = 0 \tag{6}$$

Finally, both Eqs. (3) and (6) have to be solved to determine the stationary solutions and the bifurcation points of the Navier–Stokes equations. In the following section, a new method coupling the continuation method and the homotopy technique is exposed.

3. Numerical methods

3.1. Newton's method

The way the Newton's method is applied is now presented. For a specified configuration (Ω_k), the unknowns are iteratively sought according to the following scheme:

$$\begin{cases} \lambda_{i+1} = \lambda_i + \Delta\lambda \\ U^s_{i+1} = U^s_i + \Delta U \\ V_{i+1} = V_i + \Delta V \end{cases} \tag{7}$$

where ${}^t\{\lambda_{i+1}, U^s_{i+1}, V_{i+1}\}$ is the solution at the next stage $(i + 1)$. Starting from an initial solution, let $\Delta X = {}^t\{\Delta\lambda, \Delta U, \Delta V\}$ be the correction to the solution computed at the actual stage (i). Hence, corrections $\Delta\lambda$, ΔU and ΔV are introduced in the nonlinear Eqs. (4)–(6) defining the following system:

$$\begin{cases} \mathbf{L_t}(\Delta U) = \Delta\lambda F - R^s_i \\ \mathbf{L_t}(\Delta V) = -R^v_i - \mathbf{Q}^*(\Delta U, V_i) \end{cases} \tag{8}$$

In these equations, $\mathbf{L_t}(\blacksquare) = \mathbf{L}(\blacksquare) + \mathbf{Q}^*(\blacksquare, U^s_i)$ stands for the same tangent operator applied to corrections ΔU and ΔV. Expressions R^s_i and R^v_i stand for the residues defined according to the expressions:

$$\begin{cases} R^s_i = \mathbf{L}(U^s_i) + \mathbf{Q}(U^s_i, U^s_i) - \lambda_i F \\ R^v_i = \mathbf{L}(V_i) + \mathbf{Q}^*(U^s_i, V_i) \end{cases} \tag{9}$$

As the number of unknowns, $\Delta X = {}^t\{\Delta\lambda, \Delta U, \Delta V\}$, is greater than the number of equations, an additional equation has to be introduced. The following orthogonality condition is adopted:

$$\langle V_{i+1}, V_0 \rangle = 1 \tag{10}$$

where the symbol $\langle \blacksquare, \blacksquare \rangle$ stands for the Euclidian scalar product and V_0 stands for the initial guess of the iterative which is known and regular. Finally, Eq. (8) are solved at each iteration. These two linear problems are defined with the same tangent operator $\mathbf{L_t}(\blacksquare)$ then only one matrix inversion is required at each iteration. Nevertheless, ΔU appears in the two problems and some precautions have to be taken for the numerical resolution [23,24]. Finally, Newton's iterations are performed up to the following criterion is satisfied:

$$\|R^s_i\| \leqslant \eta \quad \text{and} \quad \|R^v_i\| \leqslant \eta \tag{11}$$