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Analytical formulas for shear and bulk viscosities in relativistic gaseous mixtures with constant cross sections

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ABSTRACT

Using Mathematica package, we derive analytical closed-form expressions for the shear and the bulk viscosity coefficients in multicomponent relativistic gases with constant cross sections, being the relativistic generalization for the hard spheres model. Some of them are cumbersome and require symbolic manipulations in an algebraic package. The constant cross sections are of the elastic processes, while the inelastic (or number-changing) processes (collisions or decays) are considered only partly. As examples, we find explicit expressions of the correct single-component first-order shear viscosity coefficient and some explicit analytical results for the binary mixture. These formulas have numerous applications in approximate nonequilibrium descriptions of gases of particles or quasiparticles with averaged (temperature dependent) cross sections. In addition to this, we present formulas for collision rates and some other related formulas.

1. Introduction

The bulk and the shear viscosity coefficients are transport coefficients which enter the hydrodynamic equations, and, thus, are important for studying of nonequilibrium evolution of any thermodynamic system. In this regard, another way of dissipative nonequilibrium description can be mentioned [1,2]. In rarefied gases of particles or quasiparticles with short-range interactions the viscosities can be calculated in a perturbative regime.[1] The leading contribution can be obtained in the framework of the Boltzmann equation (BE), derivable within the BBGKY hierarchy with the well-known assumption for correlations [4] (Section 16). The BE's justification from first principles and the next-to-leading order corrections have been obtained in calculations for weakly coupled quantum field theories by the Kubo (or Green–Kubo[2]) formulas [7–10,3].[3]

The aim of this paper is to derive analytical closed-form (through special and/or elementary functions) expressions for the shear and the bulk viscosities in multicomponent relativistic gases with constant cross sections, and similarly for the collision rates and some other related quantities. Previously the single-component viscosities were obtained in [12] (we correct the shear viscos-

ity), being the relativistic generalization of the ones in the hard spheres model [4] (Sections 8 and 10).

The structure of the paper is the following. Section 2 contains some comments on cross sections, connection with the hard spheres model, most of the used designations and methodology. In Section 4 we present explicit analytical results for the single-component gas, including the ones for the nonequilibrium distribution function (DF) and some analysis for the inelastic processes. We also present some explicit expressions for the binary mixture and the collision rates (and related quantities) in the multicomponent mixture. In Section 5 we discuss significance and applications of the obtained formulas. In Section 6 we state the conclusions. Transformations of collision brackets, being the 12-dimensional integrals, which enter the viscosities, and some analytical formulas for them can be found in Appendix A.

2. Methodology

2.1. Some comments on cross sections and effective radiuses

For compatibility with previously obtained results and from practical considerations we want to introduce an effective radius r through the hard core repulsion model or the model of hard spheres. The differential scattering cross section for this model can be inferred from the problem of scattering of point particle on the spherical potential $U(r) = \infty$ if $r \leqslant a$ and $U(r) = 0$ if $r > a$ [13]. In this model the differential cross section is equal to $a^2/4$. To apply this result to the gas of hard spheres with the radius r one can notice that the scattering of any two spheres can be

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[1] In non-abelian gauge theories there are also contributions from non-perturbative effects [3].
[2] The Kubo formulas are distinguished from the Green–Kubo formulas, e.g., in [5,6].
[3] There is, however, a special important issue connected with particle number conservation/nonconservation for the bulk viscosity [11].

considered as the scattering of the point particle on the sphere of the radius $2r$, so that one should take $a = 2r$. The total cross section σ_{tot} is obtained after integration over the angles of the $r^2 d\Omega$, which results in the $\sigma_{tot} = 4\pi r^2$. For collisions of hard spheres of different radiuses one should take $a = r_k + r_l$ or replace the r on the $\frac{r_k + r_l}{2}$:

$$\sigma_{tot,kl} = \pi(r_k + r_l)^2. \tag{1}$$

The relativistic generalization of this model is the constant (not dependent on the scattering energy and angle) differential cross sections model.

The hard spheres model is classical, and connection of its cross sections to cross sections, calculated in any quantum theory, is needed. For particles, having a spin, the differential cross sections averaged over the initial spin states and summed over the final ones will be used.[4] If colliding particles are identical and their differential cross section is integrated over the momentums (or the spatial angle to get the total cross section) then it should be multiplied on the factor $\frac{1}{2}$ to cancel double counting of the momentum states. These factors are exactly the factors γ_{kl} next to the collision integrals in the BEs (17). The differential cross sections times these factors will be called (adopting the terminology mentioned in [14]) the classical differential cross sections.

2.2. The system of the BEs and its solution

The methodology in this paper goes close to the ones in the monograph [14], though with some differences (including corrections of a couple of typos) and generalizations. We find it very instructive to include compilation of some pieces of the methodology (some of which are absent in the [14]) for convenience and completeness, adding some comments and tacitly answering some questions. We use units $\hbar = c = k_B = 1$ throughout the paper by default. Conversion factors can be found, e.g., in [15]. Let's start from some definitions. We use the designations of the [14] mostly. Multi-indices k, l, m, n will be used to denote particle species with certain spin states. Indexes k', l', m', n' will be used to denote particle species without regard to their spin states (and run from 1 to the number of the particle species N') and a, b to denote conserved quantum numbers.[5] Quantifiers $\forall$ with respect to the indexes are omitted in the text where they may be needed, which will not result in a confusion. As nothing depends on spin variables, one has for every sum over the multi-indexes

$$\sum_k \cdots = \sum_{k'} g_{k'} \cdots, \tag{2}$$

where $g_{k'}$ is the spin degeneracy factor. The following assignments will be used:

$$n \equiv \sum_k n_k \equiv \sum_{k'} n_{k'}, \quad n_a \equiv \sum_k q_{ak} n_k, \quad x_k \equiv \frac{n_k}{n}, \quad x_a \equiv \frac{n_a}{n},$$

$$\hat{\mu}_k \equiv \frac{\mu_k}{T}, \quad \hat{\mu}_a \equiv \frac{\mu_a}{T}, \quad z_k \equiv \frac{m_k}{T}, \quad \pi_k^\mu \equiv \frac{p_k^\mu}{T}, \quad \tau_k \equiv \frac{p_k^\mu U_\mu}{T}, \tag{3}$$

where q_{ak} denotes values of conserved quantum numbers of the ath kind of the kth particle species. Everywhere the particle number densities are summed the spin degeneracy factor $g_{k'}$ appears and then gets absorbed into the $n_{k'}$ or the $x_{k'}$ by the definition. All other

quantities with primed and unprimed indexes do not differ, except for rates, the mean free times and the mean free paths defined in Section 4.3, the γ_{kl} commented below, the coefficients $A_{k'l'}^{rs}, C_{k'l'}^{rs}$ and, of course, quantities, whose free indexes set the indexes of the particle number densities n_k. The assignment $\int \frac{d^3 p_k}{p_k^0} \equiv \int_{p_k}$ will also be used for compactness somewhere.

The particle number flows are[6]

$$N_k^\mu = \int \frac{d^3 p_k}{(2\pi)^3 p_k^0} p_k^\mu f_k, \tag{4}$$

where the assignment $f_k(p_k) \equiv f_k$ is introduced. The energy–momentum tensor is

$$T^{\mu\nu} = \sum_k \int \frac{d^3 p_k}{(2\pi)^3 p_k^0} p_k^\mu p_k^\nu f_k. \tag{5}$$

The local equilibrium DFs are

$$f_k^{(0)} = e^{(\mu_k - p_k^\mu U_\mu)/T}, \tag{6}$$

where μ_k is the chemical potential of the kth particle species, T is the temperature and U_μ is the relativistic flow 4-velocity such that $U_\mu U^\mu = 1$ (with a frequently used consequence $U_\mu \partial_\nu U^\mu = 0$). The local equilibrium implies perturbation of the independent thermodynamic variables and the flow velocity over a *global* equilibrium (see just below) such that they can depend on the space–time coordinate x^μ. We assume chemical equilibrium, which implies that the particle number densities are equal to their global equilibrium values. We call the global equilibrium as the time-independent stationary state with the maximal entropy.[7] The global equilibrium state of an isolated system can be found by variation of the total nonequilibrium entropy functional [16] (Section 40) over the DFs with condition of the total energy and the total net charges conservation:

$$U[f] = \sum_k \int \frac{d^3 p_k d^3 x}{(2\pi)^3} f_k (1 - \ln f_k) - \sum_k \int \frac{d^3 p_k d^3 x}{(2\pi)^3} \beta p_k^0 f_k$$

$$- \sum_{a,k} \lambda_a q_{ak} \int \frac{d^3 p_k d^3 x}{(2\pi)^3} f_k, \tag{7}$$

where β, λ_a are the Lagrange coefficients. Equating the first variation to zero, one easily gets the function (6) with $U^\mu = (1, 0, 0, 0), \beta = \frac{1}{T}$ and

$$\mu_k = \sum_a q_{ak} \mu_a, \tag{8}$$

where $\mu_a = \lambda_a$ are the independent chemical potentials coupled to the conserved net charges.

With $f_k = f_k^{(0)}$, substituted in the (4) and the (5), one gets the leading contribution in the gradients expansion of the particle number flow and the energy–momentum tensor:

$$N_k^{(0)\mu} = n_k U^\mu, \tag{9}$$

$$T^{(0)\mu\nu} = \epsilon U^\mu U^\nu - P \Delta^{\mu\nu}, \tag{10}$$

where the projector

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} - U^\mu U^\nu, \tag{11}$$

is introduced. Above n_k is the ideal gas (IG) particle number density, ϵ is the IG energy density, P is the IG pressure [14] (Chapter II, Section 4). Also, the following assignments are used:

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[4] It's assumed that particle numbers of the same species but with different spin states are equal. If this were not so, then in approximation in which the spin interactions are neglected and probabilities to have certain spin states are equal the numbers of the particles with different spin states would be approximately equal in the mean free time. With equal particle numbers their DFs are equal too. This allows one to use the summed over the final states cross sections in the BEs.

[5] In systems with only the elastic collisions each particle species have their own "conserved quantum number", equal to 1.

[6] The $+, -, -, -$ metric signature is used throughout the paper.

[7] The kinetic equilibrium implies that the momentum distributions are the same as in the global equilibrium. Thus, a state of a system with both the pointwise (for the whole system) kinetic and the pointwise chemical equilibria is the global equilibrium.