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A brief note on the computation of the Bödewadt flow with Navier slip boundary conditions

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ABSTRACT

In this short communication, numerical solutions are obtained for the steady Bödewadt flow of a viscous fluid subject to partial slip boundary conditions. The resulting system of nonlinear and fully coupled similarity equations are integrated accurately by a finite difference scheme and by the Keller-box method. It is observed that slip has a prominent effect on the velocity field, reducing drastically the axial velocity and the pressure. Moreover, the torque required to maintain the disk at rest decreases for increasing values of slip.

1. Introduction

The steady laminar flow of a viscous incompressible fluid near a rotating disk, originally solved by Von Kármán [1], is one of the few problems in fluid dynamics for which the Navier–Stokes equations admit an exact solution. The twin problem arising when the fluid rotates with a uniform angular velocity at a large distance from a stationary disk, is known as the Bödewadt flow. This problem, for a viscous incompressible fluid, also admits an exact solution for the Navier–Stokes equations, subject to the conventional no-slip boundary conditions, as shown theoretically by Bödewadt [2]. The flow is characterized by the radial pressure gradient being balanced by the centrifugal forces. Fluid flows towards the axis of rotation and swepts upwards. The boundary layer, which develops on the disk, produces a secondary flow of stagnation type in the von Kármán case and of wake type in the Bödewadt problem. Batchelor [3] suggested that for large Reynolds numbers, the rotor–stator flow consists of boundary layers on each disk separated by a core of fluid rotating as a solid body. It can be seen as the connection of a Von Kármán flow along the rotor with a Bödewadt flow along the stator. Nowadays, this type of flow still receives a constant attention by the introduction of more complex and combined phenomena: heat transfer, non-Newtonian fluid [5,6], magnetic field or partial slip.

All the studies mentioned above admit no-slip condition on the walls, which is more a hypothesis than a condition deduced from any principle. Evidence of the fluid slip on a solid surface has been reported by Matthews and Hill [7]. For example, if one considers a zero-thickness disk admitting a stress-free condition on its surface and rotating around its axis, it does not modify the motion of the surrounding fluid, which would remain at rest. It confirms an intuitive result that the boundary condition on the disk plays a key role on the fluid motion. Slip condition has also some industrial relevance when the fluid is composed of emulsions, suspensions, foams or polymer solutions. In other situations where the wall surface is rough, the no-slip boundary condition also becomes impractical to apply exactly. The proper boundary condition is then well described by the general Navier's condition [8], where the amount of relative slip is proportional to the local shear stress through the slip length(es). If the characteristic scale of roughness is small compared to the boundary layer thickness, the no-slip condition may be well approximated by a partial slip condition [8]. Miklavčič and Wang [9] have considered the von Kármán swirling flow of a viscous fluid with slip boundary condition. More recently, Sherwood [10] solved the flow due to a zero-thickness disk rotating around its axis by the use of Hankel transforms. The combined effects of slip and non-Newtonian cross-viscous parameter on the rotating flows past free rotating disks have been thoroughly studied by Sahoo [11] and Sahoo and Poncet [12].

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A literature survey shows that no particular attention has been paid to the effects of slip on the Bödewadt flow of a viscous fluid. The present work is devoted to study the effects of slip on the steady Bödewadt flow of a viscous fluid. A second order finite difference method and an effective Keller box method are used to solve the fully coupled and highly nonlinear differential equations. The present paper is a step forward in the computation of the rotor–stator flow with partial slip effects to establish reference solutions for numerical benchmarks.

2. Formulation of the problem

One considers a viscous fluid occupying the space $z > 0$ over an infinite stationary disk, which coincides with $z = 0$. The motion is due to the rotation of the fluid like a rigid body with constant rotation rate Ω at a large distance from the disk. One shall assume that the principal directions of the roughness are the radial and azimuthal, i.e. a concentrically grooved disk [9], but the results could also apply to the case of a randomly rough disk. The flow is described in the cylindrical polar coordinates (r, ϕ, z) with the rotational symmetry, $\frac{\partial}{\partial\phi} \equiv 0$. Let $\mathbf{V} = (u, v, w)$ be the fluid velocity vector. Considering the usual boundary layer approximations and using the similarity transform [1]:

$$u = r\Omega F(\zeta), \quad v = r\Omega G(\zeta), \quad w = \sqrt{\Omega\nu}H(\zeta), \quad z = \sqrt{\frac{\nu}{\Omega}}\zeta, \quad p - p_\infty = -\rho\nu\Omega P \tag{1}$$

the equations of continuity and motion take the form [2,4]:

$$\frac{dH}{d\zeta} + 2F = 0, \tag{2}$$

$$\frac{d^2F}{d\zeta^2} - H\frac{dF}{d\zeta} - F^2 + G^2 = 1, \tag{3}$$

$$\frac{d^2G}{d\zeta^2} - H\frac{dG}{d\zeta} - 2FG = 0, \tag{4}$$

$$\frac{dP}{d\zeta} - H\frac{dH}{d\zeta} + \frac{d^2H}{d\zeta^2} = 0 \tag{5}$$

The no-slip boundary conditions in terms of similarity variables become,

$$\begin{aligned} &\zeta = 0: \quad F = 0, \quad G = 0, \quad H = 0, \\ &\zeta \to \infty: \quad F \to 0, \quad G \to 1, \quad P \to 0. \end{aligned} \tag{6}$$

A generalization of the Navier's partial slip condition [8,9] gives, in the radial and azimuthal directions:

$$u|_{z=0} = \lambda_1 \overline{\tau}_{rz}|_{z=0} \tag{7}$$

$$v|_{z=0} = \lambda_2 \overline{\tau}_{\phi z}|_{z=0} \tag{8}$$

where λ_1, λ_2 are the slip coefficients, and $\overline{\tau}_{rz}, \overline{\tau}_{\phi z}$ are the physical components of the stress tensor. One defines the dimensionless slip coefficients as:

$$\lambda = \lambda_1\sqrt{\frac{\Omega}{\nu}}\mu, \qquad \eta = \lambda_2\sqrt{\frac{\Omega}{\nu}}\mu. \tag{9}$$

With the help of the transformations (1), the corresponding partial slip boundary conditions (7) and (8) become:

$$F(0) = \lambda F'(0), \quad G(0) = \eta G'(0), \quad H(0) = 0, \tag{10}$$

$$F(\infty) \to 0, \quad G(\infty) \to 1, \quad P(\infty) \to 0.$$

3. Finite difference solution

The finite difference method (FDM) has been used to solve the system of coupled nonlinear Eqs. (2)–(5) subject to the slip boundary conditions (10). The semi-infinite domain $[0, \infty)$ is replaced by a finite domain $[0, \zeta_\infty)$, with ζ_∞ sufficiently large so that the numerical solution closely approximates the terminal boundary conditions. One approximates the functions and their derivatives by their finite difference counterparts to solve a sequence of linear systems.

1. One solves:

$$F'' - H^{(k)}F' = (F^{(k)})^2 - (G^{(k)})^2 + 1 \tag{11}$$

using the derivative boundary conditions (10) and denotes the solution of (11) as $\widetilde{F}^{(k+1)}$. To obtain convergence, one defines $F^{(k+1)}$ by the following smoothing formula:

$$F^{(k+1)} = \alpha_1\widetilde{F}^{(k+1)} + (1 - \alpha_1)\widetilde{F}^{(k)}, \qquad 0 \leqslant \alpha_1 \leqslant 1 \tag{12}$$

2. The same procedure is successively used for the G and H components and then for the pressure P:

$$G'' - H^{(k)}G' = 2F^{(k+1)}G^{(k)} \tag{13}$$

$$G^{(k+1)} = \alpha_2\widetilde{G}^{(k+1)} + (1 - \alpha_2)\widetilde{G}^{(k)}, \qquad 0 \leqslant \alpha_2 \leqslant 1 \tag{14}$$

$$H' = -2F^{(k+1)} \tag{15}$$

$$H^{(k+1)} = \alpha_3\widetilde{H}^{(k+1)} + (1 - \alpha_3)\widetilde{H}^{(k)}, \qquad 0 \leqslant \alpha_3 \leqslant 1 \tag{16}$$

$$P' - 2F' = -2H^{(k+1)}F^{(k+1)} \tag{17}$$

$$P^{(k+1)} = \alpha_4\widetilde{P}^{(k+1)} + (1 - \alpha_4)\widetilde{P}^{(k)}, \qquad 0 \leqslant \alpha_4 \leqslant 1 \tag{18}$$

3. The iterations start with suitable initial guesses $F^{(0)}, G^{(0)}$ and $H^{(0)}$, borrowed from the work by Sahoo and Poncet [12]. If $(F^{(k+1)}, F^{(k)})$, $(G^{(k+1)}, G^{(k)})$, $(H^{(k+1)}, H^{(k)})$ and $(P^{(k+1)}, P^{(k)})$ are close enough to each other, one stops, otherwise one sets $k = k + 1$ and goes back to step 1.

In order to solve the above system of equations by finite difference method, we introduce a grid in $0 \leqslant \zeta \leqslant \zeta_\infty$ by dividing it into n equal parts with a mesh size h equal to 0.01. It has been verified that this value guarantees a grid independent solution. One approximates the derivatives by:

$$F'(\zeta_i) = \frac{F_{i+1} - F_{i-1}}{2h}, \qquad F''(\zeta_i) = \frac{F_{i+1} - 2F_i + F_{i-1}}{h^2}, \quad i = 1, 2, \ldots n - 1 \tag{19}$$

In order to obtain a diagonally dominant linear algebraic system for Eqs. (11) and (13), F' and G' are discretized by backward difference approximations as $H_i^{(k+1)} > 0$ for Bödewadt flow. One obtains:

$$[1 + hH_i^{(k)}]F_{i-1} + [-2 - hH_i^{(k)}]F_i + F_{i+1} = h^2[(F^{(k)})^2 - (G^{(k)})^2 + 1] \tag{20}$$

$$[1 + hH_i^{(k)}]G_{i-1} + [-2 - hH_i^{(k)}]G_i + G_{i+1} = 2h^2F_i^{(k+1)}G_i^k \tag{21}$$

Finally, Eqs. (15) and (17) are discretized by central difference approximations. The above algebraic system is solved by a generalized Gauss–Seidel method instead of a successive over relaxation method. The convergence of the generalized Gauss–Seidel method for the above diagonally dominant system is reached after 15 iterations to achieve an accuracy of $O(10^{-6})$. The FORTRAN 90 code was compiled and run using the NIT Rourkela server composed of Dual Intel Xeon (8 Gb RAM, 4 Gbps Lan card).