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Implicit gas-kinetic BGK scheme with multigrid for 3D stationary transonic high-Reynolds number flows

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ABSTRACT

Instead of solving the Euler and Navier–Stokes equations directly, the gas-kinetic BGK scheme based on the Boltzmann equation has been developed and attracted more and more attentions since the early 1990s. It shows high accuracy and robustness for a wide range of flow regimes. But an obvious disadvantage of the BGK scheme is the low computational efficiency, in particular for multidimensional problems. Till now it has not been widely used as a practical tool for science and engineering applications. To overcome this drawback, in this paper some acceleration techniques, including local time stepping, implicit LU-SGS method, and multigrid strategy, are implemented into the original BGK approach and the new scheme is applied to study 3D steady transonic viscous flows. Numerical results show the significant speed up of the scheme to capture the steady state solution.

1. Introduction

Transonic aerodynamics plays an important role in the operation of long range aircrafts [1]. The cruise speed of most civil airplanes is in the transonic regime. The physical natures associated with transonic flow, including shock wave, discontinuity, transition from laminar to turbulent flow, etc., are still not fully understood. Mathematically, the governing equations are nonlinear and hybrid, which show elliptic feature in the subsonic region and hyperbolic property in the supersonic parts. Thus it is hard to find analytical solutions to predict the flow patterns and then to guide the airplane designing. Following the pioneering work of Murman and Cole [2] in the early 1970s, many numerical methods for transonic aerodynamics have been developed, from the small disturbance equation to the full potential equation, the Euler equations, and the Navier–Stokes equations. Some brief reviews can be found in [3,4]. With the ever-increasing high performance of super-computer and sophisticated software, Computational Fluid Dynamics (CFD) has been considered as a convenient and useful tool for aerodynamic analysis and designing. However, to simulate high-Reynolds number flows or around complex configurations flows is still a challenge [5]. The main difficulty is not the vast need of computing resources, but the physical modeling and numerical approaches.

Gas-kinetic BGK scheme (GKS) [6,7] has been developed for compressible flows since the early 1990s. Since then, many progress has been obtained on the validation, analysis and improvement of the GKS during the last two decades. There are numerous literatures on this topic. Xu [8] compared the gas-kinetic scheme with the Godunov method on the accuracy, efficiency, robustness, and claimed that the BGK method is more physical than the latter. Ohwada [9] explained how to construct the kinetic scheme for evolutionary equations and investigated the theoretical background and numerical errors of the scheme. Xu et al. [10] simulated hypersonic viscous flows by a multidimensional BGK flow solver with implicit LU-SGS method. Su et al. [11] and Xu et al. [12] extended the GKS to low Mach number flows. May et al. [13] improved the efficiency and convergence of the BGK scheme and presented a simplified scheme to reduce the CPU time. Issues with respect to the construction of high-order scheme [14], the extension to rarefied flows [15] and MHD [16] have also been published. Based on its constructive modeling, the GKS has a sound physical basis and can be employed as a robust and reliable flow solver to simulate problems from low Mach number flows to hypersonic flows.

Nevertheless the GKS has an obvious drawback in terms of its weak computational efficiency and expensive computational cost, which limits its practical applications. Kim et al. [17] stated that in comparison with the Riemann solver, the ratio of CPU time between convective upwind and split pressure (CUSP) scheme and standard BGK scheme per grid point per step is CUSP:BGK = 1:1.908. To overcome this disadvantage, in this paper we will present an implicit multigrid BGK scheme and extend it to solve 3D high-Reynolds number flow problems.

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2. Numerical method

2.1. Gas-kinetic BGK solver

After introducing BGK [18] model into 3D Boltzmann equation, in the absence of external force term, we get:

$$\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z}=\frac{\partial f}{\partial t}+u_\beta\frac{\partial f}{\partial x_\beta}=-\frac{f-g}{\tau}, \tag{1}$$

where f is the gas distribution function and g is the equilibrium state. τ is the relaxation time. Both f and g are functions of time t, particle velocities $u_\beta=(u,\ v,\ w)$ $(\beta=1,\ 2,\ 3)$, space $x_\beta=(x,\ y,\ z)$ and internal variable ξ. The equilibrium state g is Maxwellian

$$g=\rho\left(\frac{\lambda}{\pi}\right)^{\frac{K+3}{2}}e^{-\lambda\left((u-U)^2+(v-V)^2+(w-W)^2+\xi^2\right)},$$

where ρ is the density, $\lambda=1/2RT$ with the gas constant R and the temperature T. K denotes the number of internal freedom of ξ and equals to $(5-3\gamma)/(\gamma-1)$, here γ is the ratio of specific heats. $(U,\ V,\ W)$ are the macroscopic velocities along x-, y- and z-directions. And $\xi^2=\xi_1\xi_1+\xi_2\xi_2+\cdots+\xi_K\xi_K$.

Details on derivation from Eq. (1) to the Euler and Navier–Stokes equations can be found in [8].

Once the gas distribution is known, the macroscopic flow variables and their fluxes (including convection term and viscous term) can be obtained by the following integrals:

$$\mathbf{Q}=\begin{pmatrix}\rho\\ \rho U\\ \rho V\\ \rho W\\ \rho E\end{pmatrix}=\int f\psi_\alpha d\Xi,\quad \alpha=1,2,3,4,5, \tag{2}$$

$$\mathbf{F}_\beta=\begin{pmatrix}F_\rho\\ F_{\rho U}\\ F_{\rho V}\\ F_{\rho W}\\ F_{\rho E}\end{pmatrix}_\beta=\int u_\beta f\psi_\alpha d\Xi, \tag{3}$$

where $d\Xi=du\,dv\,dw\,d\xi$ with $d\xi=d\xi_1d\xi_2\cdots d\xi_K$, ψ_α are the collision invariants $\psi_\alpha=\left(1,u,v,w,\frac{1}{2}(u^2+v^2+w^2+\xi^2)\right)^T$, and E is the total energy per unit mass. According to the conservation laws, the right hand side of Eq. (1) at any point in space and time should satisfy the compatibility conditions:

$$\int(g-f)\psi_\alpha d\Xi=0. \tag{4}$$

To simulate multidimensional problems, the directional-splitting method has been adopted. Here, the x-directional equation, for example, is taken:

$$\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}=-\frac{f-g}{\tau}. \tag{5}$$

Following Xu's [7] suggestion, the initial gas distribution f_0 at control volume (CV) interface $x_{i+1/2}$ can be expressed as

$$f_0=\begin{cases}g^l[1+a^l(x-x_{i+1/2})-\tau(a^lu+A^l)], & x<x_{i+1/2},\\ g^r[1+a^r(x-x_{i+1/2})-\tau(a^ru+A^r)], & x>x_{i+1/2},\end{cases} \tag{6}$$

here the indexes for the other two directions $(j,\ k)$ are omitted. The superscripts 'l' and 'r' denote the left- and right-side of the interface $x_{i+1/2}$. a is the spatial slope of f, which has the form as:

$$a=a_1+a_2u+a_3v+a_4w+a_5\frac{u^2+v^2+w^2+\xi^2}{2}=a_\alpha\psi_\alpha, \tag{7}$$

and can be determined through the relationships

$$\begin{cases}\mathbf{Q}^l=\int g^l\psi_\alpha d\Xi,\\ \mathbf{Q}^r=\int g^r\psi_\alpha d\Xi,\end{cases} \tag{8}$$

$$\begin{cases}\frac{\partial\mathbf{Q}^l}{\partial x}=\frac{\mathbf{Q}^l_{i+1/2}-\mathbf{Q}_i}{\Delta x^l}=\int g^la^l\psi_\alpha d\Xi,\\ \frac{\partial\mathbf{Q}^r}{\partial x}=\frac{\mathbf{Q}_{i+1}-\mathbf{Q}^r_{i+1/2}}{\Delta x^r}=\int g^ra^r\psi_\alpha d\Xi,\end{cases} \tag{9}$$

where Δx is the distance from the CV center to the interface.

To avoid the numerical oscillation close to the interface, the conservation flow variables $\mathbf{Q}^l_{i+1/2}$ and $\mathbf{Q}^r_{i+1/2}$ are reconstructed by MUSCL interpolation [19]:

$$\begin{cases}\mathbf{Q}^l_{i+1/2}=\mathbf{Q}_i+\frac{1}{2}\varphi(\mathbf{Q}_{i+1}-\mathbf{Q}_i,\mathbf{Q}_i-\mathbf{Q}_{i-1}),\\ \mathbf{Q}^r_{i+1/2}=\mathbf{Q}_{i+1}-\frac{1}{2}\varphi(\mathbf{Q}_{i+1}-\mathbf{Q}_i,\mathbf{Q}_{i+2}-\mathbf{Q}_{i+1}),\end{cases} \tag{10}$$

where φ is a limiter function $\varphi(y_1,y_2)=\frac{y_1(y_2^2+\epsilon)+y_2(y_1^2+\epsilon)}{y_1^2+y_2^2+2\epsilon}$. The parameter ϵ is a small constant with the order being proportional to the local grid scale.

On the fact that the non-equilibrium terms of Eq. (6) have no contributions to the conservative variables, we know:

$$\begin{cases}\int g^l(a^lu+A^l)\psi_\alpha d\Xi=0,\\ \int g^r(a^ru+A^r)\psi_\alpha d\Xi=0.\end{cases} \tag{11}$$

By solving Eq. (11) the temporal slope A will be obtained.

The equilibrium state g is assumed to have two slopes too [7]

$$\begin{aligned}g=g_0[1+(1-H[x-x_{i+1/2}])\bar a^l(x-x_{i+1/2})\\ +H[x-x_{i+1/2}]\bar a^r(x-x_{i+1/2})+\overline{A}t],\end{aligned} \tag{12}$$

where $H[x]$ is the Heaviside function

$$H[x]=\begin{cases}1, & x\geqslant 0,\\ 0, & x<0.\end{cases} \tag{13}$$

$\bar a$ has the similar form of a: $\bar a=\bar a_\alpha\psi_\alpha$ and can be determined by the relation of

$$\begin{cases}\frac{\mathbf{Q}^0_{i+1/2}-\mathbf{Q}_i}{\Delta x^l}=\int g_0\bar a^l\psi_\alpha d\Xi,\\ \frac{\mathbf{Q}_{i+1}-\mathbf{Q}^0_{i+1/2}}{\Delta x^r}=\int g_0\bar a^r\psi_\alpha d\Xi,\end{cases} \tag{14}$$

where $\mathbf{Q}^0_{i+1/2}$ is the vector of the initial conservative values at CV interface $x_{i+1/2}$, which can be calculated by

$$\mathbf{Q}^0_{i+1/2}=\int g_0\psi_\alpha d\Xi=\int\int_{u>0}g^l\psi_\alpha d\Xi+\int\int_{u<0}g^r\psi_\alpha d\Xi. \tag{15}$$

So far, the only unknown coefficient is $\overline{A}$.

The general solution f of BGK Eq. (1) can be expressed as:

$$\begin{aligned}f(x_{i+1/2},t,u_\beta,\xi)=\frac{1}{\tau}\int_0^t g(x_{i+1/2}-u(t-t'),t',u_\beta,\xi)e^{-(t-t')/\tau}dt'\\ +e^{-t/\tau}f_0(x_{i+1/2}-ut,0,u_\beta,\xi),\end{aligned} \tag{16}$$

Substituting Eqs. (6) and (12) into the above formula, after integrating, we get the gas distribution function f:

$$\begin{aligned}f(x_{i+1/2},t,u_\beta,\xi)=(1-e^{-t/\tau})g_0+\tau(t/\tau-1+e^{-t/\tau})\overline{A}g_0\\ +(\tau(-1+e^{-t/\tau})+te^{-t/\tau})(\bar a^lH[u]+\bar a^r(1\\ -H[u]))ug_0-\tau e^{-t/\tau}(A^lH[u]g^l+A^r(1\\ -H[u])g^r)+e^{-t/\tau}((1-u(t+\tau)a^l)H[u]g^l\\ +(1-u(t+\tau)a^r)(1-H[u])g^r).\end{aligned} \tag{17}$$

Then substituting Eqs. (12) and (17) into the compatibility conditions (4), a simplified algorithm for $\overline{A}$ is given as below [20,21]