Computers & Fluids 76 (2013) 73–85

Contents lists available at SciVerse ScienceDirect

Computers & Fluids

journal homepage: www.elsevier.com/locate/compfluid

Computation of Hopf bifurcations coupling reduced order models and the asymptotic numerical method

J. Heyman [a,1], G. Girault [a,b], Y. Guevel [a], C. Allery [c], A. Hamdouni [c], J.M. Cadou [a,*]

[a] Laboratoire d'Ingénierie des Matériaux de Bretagne, Université Européenne de Bretagne, Université de Bretagne Sud, Rue de Saint Maudé, B.P. 92116, 56321 Lorient Cedex, France
[b] Centre de recherche des Écoles de Saint-Cyr Coëtquidan, Écoles de Coëtquidan, 56381 Guer Cedex, France
[c] LASIE, Université de la Rochelle, Pôle Science et Technologie, Avenue Michel Crépeau, 17042 La Rochelle Cedex 1, France

ARTICLE INFO

Article history:
Received 30 August 2012
Received in revised form 7 January 2013
Accepted 4 February 2013
Available online 16 February 2013

Keywords:
Hopf bifurcation points
Reduced order model
Asymptotic numerical method
Fluid mechanics

ABSTRACT

This work deals with the computation of Hopf bifurcation points in the framework of two-dimensional fluid flows. These bifurcation points are determined by using a Hybrid method [1] which associates an indicator curve and a Newton method. The indicator provides initial values for the Newton method. As the calculus of this indicator is time consuming, we suggest using an algorithm to save computational time. This algorithm alternates reduced order and full size step resolution which are all carried out by using a pertubation method. Hence, the computed vectors on the full size problem are used to define the reduced order model. As the low-dimensional model has a finite validity range, we propose a simple criterion which makes it possible to know when the basis has to be updated. The latter phase is carried out by going through a new full step which permits to build a new basis and, thus, compute a supplementary part of the indicator curve. Some numerical tests, such as the classical lid-driven cavity or the flow in a channel, permit to fix the optimum values of the parameters for the proposed method. The objective of this study is to save computational time without modifying the performance of the Hybrid method initially introduced in Ref. [1]. These numerical methods are applied to 2D fluid flows (flow in a channel and the 2D lid-driven cavity). Our conclusions, therefore, hold only for these kinds of problem.

1. Introduction

A Hopf's bifurcation is an instability of the fluid flow which is characterized by the transition from a stationary state towards an instationary one. From a mathematical point of view, a Hopf's bifurcation appears when a complex conjugate pair of eigenvalues of the linearized Jacobian matrix crosses the imaginary axis of the complex plane.

Usually, such instabilities are numerically computed by means of the so-called direct methods and the indirect ones with the monitoring of an indicator.

The principle of the direct method consists in iteratively computing the solution of a nonlinear algebraic system which corresponds to a Hopf bifurcation point [4–6]. Unfortunately, the convergence of the algorithm depends on the choice of the initial value.

The indirect method is based on the track of an indicator which has the property of being null at a bifurcation point. The latter consists, for example, in computing the eigenvalues of the Jacobian matrix, and in determining when a pair of complex conjugated values crosses the imaginary axis, see Refs. [2,3]. In [7], Cadou et al. use an indirect method to determine the Hopf bifurcation for the 2D academic problems of the flow around a cylinder and the flow in a lid-driven cavity. In fact, they introduce a bifurcation indicator which has the property of being null at the singular points. To avoid large computing times, the computation of this bifurcation indicator is done with a perturbation method. Whereas it gives accurate values of bifurcation points, this method requires a lot of calculi, and is not automatic.

To circumvent this drawback, Brezillon et al. [1] propose to couple two methods (direct and indirect ones) resulting in a hybrid algorithm. The idea of this hybrid method is that the initial guesses of the Newton algorithm are determined by an indicator calculation. The numerical results show that the indicator calculation provides several initial values but some candidates do not lead to the convergence of the Newton's algorithm, resulting in a large amount of CPU time without ensuring the convergence of the hybrid method.

Recently, Girault et al. [8] propose to improve the robustness of the hybrid method by automatically determining the minima of the indicator curve, and using them as initial values for the Newton algorithm. In the case of the 2D lid driven cavity, the results show that the method is efficient. Nevertheless, this algorithm still

* Corresponding author. Tel.: +33 2 97 87 45 74.
E-mail address: jean-marc.cadou@univ-ubs.fr (J.M. Cadou).
[1] Current address: École Polytechnique Fédérale de Lausanne, Laboratoire d'Hydraulique Environnementale, Station 18, 1015 Lausanne, Switzerland.

0045-7930/$ - see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.compfluid.2013.02.001

requires long CPU times mainly dedicated to the calculation of the indicator because of the size of the manipulated Jacobian matrix. Previous studies, [1,8], have shown that the calculation of the indicator is the step that consumes the most CPU time in the hybrid method. The idea of the present study is to propose an efficient numerical method allowing for a faster calculation of the indicator.

A means of decreasing these CPU times is either the use of a specific linear solver [9,10] or that of reduced order models. The main objective of such methods is to replace a large fully dicretized model with a reduced model describing correctly the dynamic behavior and preserving fundamental properties of the full model. Hence, the choice of the reduction technique is important. Today, the POD is one of the most widely used techniques of model order reduction. Introduced for the first time by Lumley [11] for the study of turbulent flows, the POD consists in a linear decomposition yielding a physical and orthogonal spatial basis in which dimensions are lower than the initial model. The reduced order models are then obtained by projecting the full model onto this POD basis. The main drawback of the POD is that it requires computations of the unkown fields, in the full size problem, to build the snapshots needed for determination of the reduced basis. This approach was used by Cazemier et al. [12] to compute Hopf bifurcations in 2D lid-driven cavities. They carry out a first time-dependent simulation of the Navier–Stokes equations for a large Reynolds number. Next, they use these results to build, with the POD, a reduced eigenvalues problem in order to determine precise Hopf bifurcation points. In the case of the hybrid method, this reduction technique is not the best way to proceed. Indeed, as shown in Ref. [8], a single computation of the indicator can provide the Newton method with a lot of initial guesses, which can lead sometimes to 4 or 5 Hopf bifurcation points. A POD analysis can then be performed using this first computation although results will probably be the same as the ones obtained with the initial calculi. So, the benefit of using a reduced model is nullified by the fact that it requires a first computation which can be time consuming and not useful for the determination of bifurcation points.

A reduced technique has been recently proposed in [13] and applied to define a linear solver [10] or to study nonlinear vibrations of plates [14]. In this technique, the vectors computed in the first steps of the perturbation method are used to build a basis which permits to determine the other part of the nonlinear solution. In the case of the linear solver, a preconditioning technique is added to the reduction method with a view to avoiding a lot of basis modifications. It means that the basis is the same for almost all the computations. In the framework of nonlinear vibrations of plates [14], as the nonlinear curves do not evolve a lot all through the computations, a single basis computation is necessary to compute almost all the nonlinear solution curves. For the hybrid method, as shown in references [1,8], the indicator curves which depend on the angular frequency are very nonlinear. A single basis evaluation is therefore not sufficient to determine the entire response indicator curve. So the basis has to be upgraded all along the indicator computation. We propose to alternate between full size problem resolution and reduced order models steps. A full size computation is performed when the reduced order solutions do not verify a simple residual criterion. With this full size computation, a new basis is defined and permits then to carry out additional calculi of the indicator on the reduced order problem. As the computation on the full size model is very time consuming, mainly due to the fine spatial discretization, the point is to limit these full size computational steps.

The paper is organized as follows. Section 2 is devoted to theoretical aspects recalling the governing equations for an incompressible viscous flow, and the stability analysis. Section 3 presents the model reduction technique. Some numerical results related to the academic problems of the lid-driven cavity and the flow in a channel are given in Section 4, and show the relevancy of the proposed method.

2. Elements of theory

2.1. Governing equations

In this study, we consider the movement of a viscous incompressible flow described by the following Navier Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\nabla^2\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{\nabla p}{\rho} = 0 \quad \text{in } (\Omega) \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } (\Omega) \tag{2}$$

$$\mathbf{u} = \lambda\mathbf{u}_d \quad \text{on } (\partial_{\mathbf{u}}\Omega) \tag{3}$$

In these equations, Ω and $\partial_{\mathbf{u}}\Omega$ are the fluid domain and the boundary surface where velocity is imposed. The symbols $\mathbf{u}$, p, ρ, ν stand, respectively, for the velocity field, the pressure, the density and the kinematic viscosity of the fluid. The boundary condition imposes a velocity field of intensity λ which is linked to the Reynolds Number defined by $Re = \lambda|\mathbf{u}_d|L/\nu$ with L being a geometrical reference length. For each numerical example, this Reynolds number will be precisely defined and is the bifurcation parameter used in this study.

The weak formulation associated to equations [17] is written:

$$M(\dot{\mathbf{U}}) + L(\mathbf{U}) + Q(\mathbf{U},\mathbf{U}) - \lambda F = 0 \tag{4}$$

where M is the mass matrix, L and Q are linear and quadratic operators: L contains the pressure and the diffusion terms while the convective terms are contained in Q. The vector $\mathbf{U}$ is a concatenated vector composed of the velocity $\mathbf{u}$ and the pressure p. The term λF can be considered as an external load vector created by the boundary condition on $\partial_{\mathbf{u}}\Omega$. One can refer to [17] for a complete presentation of all these operators.

2.2. Stability analysis

The stability of the flow is studied by introducing a perturbation $\Delta U(\mathbf{x},t)$ of the stationary term U^S. This perturbation can be considered as a product of the spatial term $V(\mathbf{x})$ by the temporal term $e^{i\omega t}$:

$$\Delta U(\mathbf{x},t) = V(\mathbf{x})e^{i\omega t} \tag{5}$$

In expression (5), ω designates the pulsation of the periodic flow, and $V(\mathbf{x})$ stands for the complex mode of perturbation.

Introducing expression (5) into Eq. (4), and neglecting quadratic and higher order terms in V, it becomes the following linear system:

$$\begin{cases} L(U^S) + Q(U^S,U^S) - \lambda F^S = 0 \quad \text{in} \quad \Omega \\ L(V) + Q(V,U^S) + Q(U^S,V) + i\omega M(V) = 0 \quad \text{in} \quad \Omega \\ V_{\mathbf{u}} = 0 \quad \text{on} \quad \partial_{\mathbf{u}}\Omega \end{cases} \tag{6}$$

A Hopf's bifurcation corresponds to a vector $\Lambda = \{U^S, V, \lambda, \omega\}$ which is the solution to Eq. (6). In the latter equations, $V_{\mathbf{u}}$ stands for the velocity part of the concatenated vector V on the boundary $\partial_{\mathbf{u}}\Omega$ where a velocity $\mathbf{u}_d$ is imposed. Finally, the previous system is written under the following form:

$$R(\Lambda) = 0 \quad \text{in } \Omega \tag{7}$$

2.3. Bifurcation indicator

This section is devoted to the presentation of the bifurcation indicator and of the asymptotic numerical method applied to the search of Hopf's bifurcation points. All the elements have already been presented in [7,8] and the main ideas are outlined in this paper.