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Numerical simulation of rotation dominated linear shallow water flows using finite volume methods and fourth order Adams scheme

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ABSTRACT

In this paper, we study the performance of some finite volume schemes for linear shallow water equations on a rotating frame. It is shown here that some well-known upwind schemes, which perform well for gravity waves, lead to a high level of damping or numerical oscillation for Rossby waves. We present a modified five-point upwind finite volume scheme which leads to a low level of numerical diffusion and oscillation for Rossby waves. The method uses a high-order upwind method for the calculation of the numerical flux and a fourth-order Adams method for time integration of the equations and is considerably more efficient than the fourth-order Runge–Kutta method that is usually used for temporal integration of shallow water equations in the presence of the Coriolis term. In the method proposed here, the Coriolis term is treated analytically in two stages: before and after calculation of computational fluxes. It is shown that the energy dissipation of the proposed method is considerably less than other upwind methods that are widely used, such as the third-order upwind method.

1. Introduction

Shallow flows exist in many cases in nature. In these flows, horizontal scales are much larger than the vertical ones, the pressure is close to hydrostatic, and the velocity profile has small changes throughout the depth (e.g., [20]). They can be found in most shallow lakes, reservoirs, rivers, and oceans, and several types of atmospheric flow.

In the past, considerable research has been performed on the numerical simulation of shallow flows, and several high-resolution schemes have been proposed for numerical solution of shallow water equations (e.g., [23]). Finite volume schemes are a class of numerical methods which are well known to inherently conserve mass and momentum (in the absence of source terms). This is because these schemes are in the flux form, and the mass going out of a computational control volume directly enters the adjacent cell. Therefore, the mass, and similarly the momentum, are globally conserved; e.g., when periodic boundary conditions are imposed. Upwind finite volume schemes have been the most popular finite volume methods for hyperbolic systems in the past because they can capture discontinuities within a few computational cells with a low level of numerical diffusion and oscillation. The critical stage in finite volume schemes is the computation of the numerical flux, and numerous schemes have been developed for estimation of this.

In upwind finite volume methods, the characteristics of the hyperbolic system are used to calculate the numerical flux. For a scalar advection equation, this simply becomes a discretization of the equation in the flow direction, i.e., using the values of the cells in the upstream direction. However, when a system of equations is considered, the flow direction is not the only parameter in the computation of numerical flux. In this case, the flux vector is decomposed on the basis of the eigenvectors, and then each component is discretized using an upwind method, i.e., based on the direction of the corresponding eigenvalue.

Shallow water equations, in the absence of viscous terms, can be considered to be a hyperbolic system. Extensive studies have been conducted to improve the performance of upwind schemes for the shallow water system, especially in the cases where gravity effects are dominant, such as shock waves. Upwind finite volume schemes have been successfully developed for the simulation of some challenging problems concerning shallow waters, such as supercritical flows over spillways and dam break flows [17]. Simulation of such problems without numerical oscillation is beyond the capability of most other existing schemes. However, a major problem of upwind methods with shallow water systems is an imbalance between the flux and source terms at the discrete level, and extensive studies have focused on this issue (e.g., [13,14,18]).

On the other hand, shallow water systems also exist in many other natural circumstances, including oceanic and atmospheric circulations (e.g., [20]). In such cases, in addition to the convective terms, the Coriolis effect also plays an essential role in the flow

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pattern. Indeed, the shallow water system allows for several waves, such as gravity, inertio-gravity, and Rossby waves, which have completely different structures. For example, the phase speeds of these waves are largely different; gravity waves propagate with a relatively fast speed, while Rossby waves are very slow. Accurate simulation of all waves is a delicate problem for most available schemes. In general, upwind schemes perform very well for gravity and shock waves, but as shown in Mohammadian and Le Roux [15], they are too diffusive for Rossby waves. On the other hand, centered schemes perform better for slow modes such as Rossby waves, but they present poor performance for gravity and shock waves. The application of upwind schemes for rotation-dominated cases still needs to be studied. For example, the phase speed of Rossby waves and their damping due to numerical schemes is an important issue in oceanic and atmospheric circulation modeling [12]. Indeed, most energy transfer in the ocean and atmosphere is performed by Rossby waves. Therefore, damping of Rossby waves beyond a certain level is undesirable and leads to erroneous results. On the other hand, in oceanic and atmospheric circulations, small-scale and fast gravity waves are mainly considered as noise which do not play an essential role in energy transfer and general circulation. Therefore, in order to increase stability, it is usually desirable to damp noises, which is perfectly done by upwind schemes. Thus, a desirable scheme for the purpose of simulation of general circulation should present a low level of damping of the Rossby waves, while damping noises.

A large number of studies have been conducted in the past to evaluate the performance of various schemes for Rossby waves [21,2,4,22,5,9,10,16]. Le Roux and Carey [8] studied the least-square finite volume method and showed that it leads to a high level of damping compared to the Galerkin scheme for the case of gravity and Rossby waves, and they concluded that while the staggered grid presents better results in terms of damping, this scheme should be used with care, particularly for long term simulations due to its numerical damping. Mohammadian and Le Roux [15] performed a one-dimensional Fourier analysis for a class of upwind schemes, and concluded that the κ scheme along with the second-order Runge Kutta method, while having a good performance for gravity waves, is not a good choice for Rossby waves. However, a "universal scheme" is not available with optimal performance for all flow scales, and therefore the employed numerical method should be chosen based on the targeted flow regime. On the other hand, noting that the Coriolis term is a source term, an imbalance between the flux and source terms in the upwind method arises in the case of Rossby waves as well, and makes their simulation with upwind methods even more complicated.

In this paper, we study the performance of some finite volume schemes for linear SW equations on a rotating frame. As we will show, the selected upwind schemes lead to either a high level of damping for Rossby waves, or numerical oscillations due to an imbalance between the flux and source terms. We then present a modified high-order upwind scheme which leads to accurate results for both Rossby and gravity waves. It is also shown that an analytical solution for the Coriolis term leads to good results for Rossby waves. Furthermore, we show that the fourth-order Adams method, which has been rarely if ever used with upwind schemes for shallow water equations, is a good and computationally efficient alternative to the commonly used high-order Runge–Kutta scheme for time integration. As mentioned before, the focus of this paper is on the Coriolis term. Since the Coriolis term remains linear in the full nonlinear system, the scheme presented here for the Coriolis term may be also applicable to nonlinear cases. The treatment of other terms such as bed friction and topography is a separate issue and traditional methods such as upwind discretization of source terms may be used for them (see e.g. [19]). The extension of the method to nonlinear equations is currently in progress by the authors and promising preliminary results have been obtained which will be presented in a subsequent paper. However, it should be mentioned that the linear SW equations are also important because they appear in multi-scale equations developed recently for some atmospheric flows (see e.g. Khouider and Majda [6]) and their solution is a challenging issue. They also appear in spectral methods based on vertical mode decomposition in the atmosphere (see e.g. [11]).

This paper is organized as follows. In Section 2 the shallow water equations are presented. Section 3 explains the finite volume methods where the computational details for the treatment of various terms are discussed. In Section 4 some numerical results are presented to compare the proposed method with other available upwind schemes. Some concluding remarks complete the study.

2. The shallow water equations

The 2-D linear SW equations in a conservative form may be written as ([20])

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{E}}{\partial x}+\frac{\partial \mathbf{G}}{\partial y}=\mathbf{S}, \tag{1}$$

with

$$\mathbf{U}=\begin{bmatrix}\eta\\ u\\ v\end{bmatrix},\qquad \mathbf{E}=\begin{bmatrix}Hu\\ g\eta\\ 0\end{bmatrix},\qquad \mathbf{G}=\begin{bmatrix}Hv\\ 0\\ g\eta\end{bmatrix} \tag{2}$$

where η is the water surface elevation, u and v are the velocity components, g is the gravitational acceleration, and $(H+\eta)$ is the total water depth. The term $\mathbf{S}$ may include various source terms such as bed roughness, Coriolis, and topography. Since this paper concentrates on Rossby waves, the source term $\mathbf{S}$ is assumed to include the Coriolis parameter

$$\mathbf{S}=\begin{bmatrix}0\\ fv\\ -fu\end{bmatrix} \tag{3}$$

We can also write the system in the following non-conservative form

$$\frac{\partial \mathbf{U}}{\partial t}+A\frac{\partial \mathbf{U}}{\partial x}+B\frac{\partial \mathbf{U}}{\partial y}=\mathbf{S}, \tag{4}$$

where

$$A=\begin{bmatrix}0&H&0\\ g&0&0\\ 0&0&0\end{bmatrix},\quad B=\begin{bmatrix}0&0&H\\ 0&0&0\\ g&0&0\end{bmatrix} \tag{5}$$

The matrix A has three real eigenvalues, given by

$$\lambda_1=\sqrt{gH} \tag{6}$$
$$\lambda_2=-\sqrt{gH} \tag{7}$$
$$\lambda_3=0 \tag{8}$$

and the following corresponding eigenvectors:

$$\mathbf{e}_1=\begin{bmatrix}1\\ +\sqrt{g/H}\\ 0\end{bmatrix},\quad \mathbf{e}_2=\begin{bmatrix}1\\ -\sqrt{g/H}\\ 0\end{bmatrix},\quad \mathbf{e}_3=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix} \tag{9}$$

The matrix A is then decomposed as

$$A=PDP^{-1} \tag{10}$$