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Two-dimensional simulation of shallow-water waves by Lagrangian block advection

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ABSTRACT

Waves in shallow water are computed by moving blocks of water in the direction of the flow using a Lagrangian method. The mass and momentum in the displaced-and-deformed blocks after the Lagrangian advection are re-distributed back on to the Eulerian mesh to form new blocks at every increment of time. This Lagrangian block advection guarantees for positive water depth. It also prevents the occurrence of unphysical numerical oscillations. Several numerically challenging problems are considered in a series of simulations using the method. The first problem is the tracking of wetting-and-drying interface in a parabolic bowl. The second problem is the capture of depth and velocity discontinuities across the shock waves. Finally, the block advection method is applied to calculate the flood waves overtopping a meandering river. The results of the simulations are compared with the exact solutions. The convergence of Lagrangian block advection towards the exact solutions is first-order accurate in the simulations of the depth-and-velocity discontinuities.

1. Introduction

Computational stability is crucial to many engineering simulation problems including the flood waves over lands, the evolution of avalanches, the run-up of waves on beaches and the overtopping of water on levees. The computation must capture the discontinuities across the shock waves and the discontinuities at the wave fronts where the wet water meets the dry land. In the classical finite-volume simulations, the discontinuities are the source of unphysical numerical oscillations which often lead the computation to failure. A variety of ad-hoc numerical methods have been developed to manage the computational instability. Shock capture schemes [6,7,26] and flux limiters [10,14,17] have been the methods to control the unphysical numerical oscillations. The advance and the retreat of waters on dry land have been attempted with some success using the wet-and-dry threshold [13,15], the wet cell mapping [9], the artificial porosity techniques [27], the volume-of-fluid method [8,11], the Lagrangian–Eulerian algorithms [1,2,12], and the technique of the artificial viscosity [28,18].

As an alternative to the classical methods, a Lagrangian block advection (LBA) method has been developed by Tan and Chu [22,21] for one-dimensional simulations of water waves in shallow water. In the LBA simulations, the mass and momentum in the water waves are transferred by the Lagrangian advection of the blocks. The LBA method always gives positive water depth. It correctly captures the depth-and-velocity discontinuities while maintaining absolute computational stability. The method has since been applied to a number of one-dimensional (1D) water engineering problems. These include the dam-break waves [21], the collapsing bore [22,19,16], the runup and overtopping of solitary waves [23] and the runup and overtopping of the regular waves [24].

This paper will show how the LBA method is generalized for application to two-dimensional (2D) problems. A couple of 2D analytical solutions involving flow discontinuities is used as the benchmarks. The first of the 2D benchmarks is the solution for water waves in a parabolic bowl by Thacker [25]. The second of the 2D benchmarks is the solution for the shock waves by Stoker [20]. The 2D LBA simulations for these benchmark problems are carried out using progressively smaller block sizes. The convergence is verified by the comparison of the simulations with the analytical solutions. Finally, the versatility of 2D LBA method for engineering application is demonstrated by routing flood waves in a meandering river.

2. Lagrangian block advection

The Lagrangian blocks as the computational elements are defined by the dimensions of the blocks Δx^L and Δy^L and the contents such as volume and momentum in the blocks. Three separate systems of blocks for the volume, $h^L_{i,j}\Delta x^L\Delta y^L$, x-momentums, $u^L_{i,j}\bar{h}^u_{i,j}\Delta x^L\Delta y^L$ and y-momentum $v^L_{i,j}\bar{h}^v_{i,j}\Delta x^L\Delta y^L$, are employed for the LBA simulations on a staggered grid. The superscript 'L'

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distinguishes the Lagrangian variables from the Eulerian variables. Fig. 1 shows (a) the staggered grid, (b) the volume block, (c) the x-momentum block, and (d) the y-momentum block. As shown in (b–d) of the figure, the blocks initially occupy the same area as the Eulerian cell at the beginning of time step. The contents in the blocks move with the blocks as the blocks are displaced and deformed with the flow. The transfer of the block's contents to its neighboring cells is completed when the old blocks are break up along the grid lines and the new blocks are formed at the end of the Lagrangian advection time step.

Fig. 1b delineates the Lagrangian advection of the volume block. At the beginning of the Lagrangian advection at time t, the edges of the blocks match the Eulerian mesh, i.e.,

$$x^L_{i,j}(t) = x_{i,j}(t),\ y^L_{i,j}(t) = y_{i,j}(t). \tag{1}$$

At the end of the advection at time $t + \Delta t$,

$$x^L_{i,j}(t+\Delta t) = x_{i,j}(t) + \int_o^{\Delta t}\int_o^t a^{xL}_{i,j}\,dt'\,dt, \tag{2}$$

$$y^L_{i,j}(t+\Delta t) = y_{i,j}(t) + \int_o^{\Delta t}\int_o^t a^{yL}_{i,j}\,dt'\,dt. \tag{3}$$

For the waves in shallow-waters, the pressure over the depth may be assumed hydrostatic. The x- and y-components of the depth-averaged flow accelerations are given by the shallow-water equations as follows:

$$a^{xL}_{i,j} = \frac{Du^L_{i,j}}{Dt} = -g\frac{\zeta^L_{i,j} - \zeta^L_{i-1,j}}{\Delta x} + f^x_{i,j}, \tag{4}$$

$$a^{yL}_{i,j} = \frac{Dv^L_{i,j}}{Dt} = -g\frac{\zeta^L_{i,j} - \zeta^L_{i,j-1}}{\Delta y} + f^y_{i,j}, \tag{5}$$

where D/Dt = Lagrangian time-differentiation operator, $\left(u^L_{i,j}, v^L_{i,j}\right)$ = x- and y-components of the flow velocity, $\zeta^L_{i,j} = h^L_{i,j} + z^o_{i,j}$ = water surface elevation, $h^L_{i,j}$ = water depth, $z^o_{i,j}$ = channel bottom elevation, g = gravity, and $\left(f^x_{i,j}, f^y_{i,j}\right)$ = x- and y-components of the other forces such as the friction force. The integrations for the edge co-ordinates of the block, $\left(x^L_{i,j}, y^L_{i,j}\right)$, are carried out using the approximation that the accelerations $\left(a^{xL}_{i,j}, a^{yL}_{i,j}\right)$ are constant throughout the period of the Lagrangian advection from time t to $t + \Delta t$.

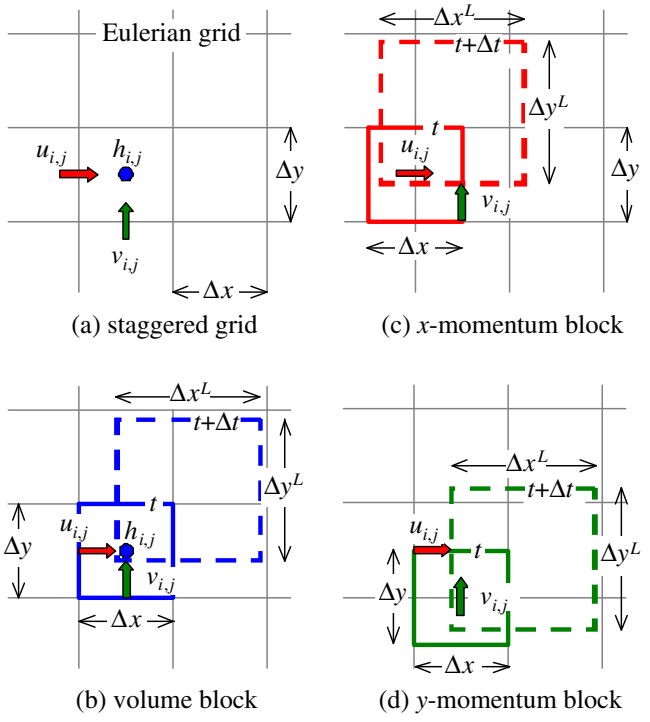

Fig. 1. (a) The staggered grid, (b) the volume blocks, (c) the x-momentum blocks, and (d) the y-momentum blocks. The solid rectangles show the blocks before the Lagrangian advection. The dashed rectangles delineate the edges of the blocks after the Lagrangian advection.

Fig. 2 shows how the area of the (i,j)-block is displaced-and-deformed from $\Delta x\Delta y$ to $\Delta x^L\Delta\ y^L$ due the Lagrangian advection. The block occupies initially the same area as the Eulerian cell. The displaced-and-deformed block may occupy an area covering as much as eight neighboring cells as shown. The displaced-and-deformed block is divided along the grid lines into portions and then re-distributed onto the Eulerian mesh to form new blocks at the end of the advection step. A block re-distribution algorithm had been developed to (i) subdivide the old block along the grid lines, (ii) re-distribute the block's content onto its neighboring cells, and (iii) re-construct the new blocks at every time increment. Using the block re-distribution algorithm, the contents in the block are transported across the grid lines from the cell to its neighboring cells. The computational time step must not be too large to cause advection beyond its neighbors. The block re-distribution algorithm was developed by Chu and Altai [4,5] based on a requirement that the displaced-and-deformed block boundary stays within the immediate neighboring cells. This requirement gives a *necessary* condition for the computational stability. Over the period of one Δt, the displacements of all block boundaries $\mathrm{Max}|u_{i,j}|\Delta t$ and $\mathrm{Max}|v_{i,j}|\Delta t$ must not exceed Δx and Δy in the x- and y-directions, respectively. Therefore, The Courant numbers Co_x and Co_y must kept below the value of unity during the computation as follows:

$$\mathrm{Co}_x = \frac{\mathrm{Max}|u_{i,j}|\Delta t}{\Delta x} < 1, \tag{6}$$

$$\mathrm{Co}_y = \frac{\mathrm{Max}|v_{i,j}|\Delta t}{\Delta y} < 1. \tag{7}$$

These necessary conditions for stability would keep the time step Δt sufficiently small so that the deformation in one advection step would not be excessive to cause *Lagrangian entanglement*. The area of the block $\Delta x^L_{i,j}\Delta y^L_{i,j}$ may become negative after the Lagrangian advection if the edge on one side of the block overtakes the other side from behind.

The third necessary condition for the computational stability is the Courant–Friedrichs–Lewy (CFL) condition for the shallow-water waves:

$$\mathrm{Co}_c = \frac{\sqrt{g\mathrm{Max}[h_{i,j}]}\Delta t}{\Delta x} < 1, \tag{8}$$

where $\sqrt{g\mathrm{Max}[h_{i,j}]}$ is the wave speed of the shallow-water waves. The time step must be selected to meet all three necessary conditions as given by the formula:

$$\Delta t = \mathrm{Min}\left\{\frac{\Delta x}{\sqrt{g\mathrm{Max}[h_{i,j}]}}, \frac{\Delta x}{\mathrm{Max}|u_{i,j}|}, \frac{\Delta y}{\mathrm{Max}|v_{i,j}|}\right\}\mathrm{Co}. \tag{9}$$

According to this formula, the time step Δt is proportional to the Courant number Co. The necessary condition for computational stability is Co < 1.

As the stability, the accuracy of the Lagrangian advection also depends on the time step size and Co. Since the value of Co = 0.2 was used in most of the computations carried out using the classical finite-volume (CFV) method, the same value Co = 0.2 is selected for the present series of LBA simulations. With this selection, the results of the present LBA simulations are comparable with the results obtained using the CFV method.