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Modelling solute transport in shallow water with the lattice Boltzmann method

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ABSTRACT

The lattice Boltzmann method is used to investigate the solute transport in shallow water flows. Shallow water equations are solved using the lattice Boltzmann equation on a D2Q9 lattice with multiple-relaxation-time (MRT-LBM) and Bhatnagar–Gross–Krook (BGK-LBM) terms separately, and the advection–diffusion equation is also solved with a LBM-BGK on a D2Q5 lattice. Three cases: open channel flow with side discharge, shallow recirculation flow and flow in a harbour are simulated to verify the described methods. Agreements between predictions and experiments are satisfactory. In side discharge flow, the reattachment length for different ratios of side discharge velocity to main channel velocity has been studied in detail. Furthermore, the performance of MRT-LBM and BGK-LBM for these three cases has been investigated. It is found that LBM-MRT has better stability and is able to satisfactorily simulate flows with higher Reynolds number. The study shows that the lattice Boltzmann method is simple and accurate for simulating solute transport in shallow water flows, and hence it can be applied to a wide range of environmental flow problems.

1. Introduction

Shallow water flows exist in rivers, channels, coastal seas, estuaries and harbours. The flows are described by shallow water equations, which are widely used in hydraulic engineering. Recently, much attention has been given to solute transport in the shallow waters such as distribution of pollution concentration and transport of suspended sediments [1–6]. Thus, prediction of the flows and related transport is important in environmental engineering.

As a relatively new numerical approach, the lattice Boltzmann method has been used successfully in various areas [7]. Its advantage in solving the shallow water equations [8–14] and advection–diffusion equation have been widely demonstrated [15–18].

The Bhatnagar–Gross–Krook (BGK) term is often applied to the most popular lattice Boltzmann method due to its great simplicity [19,20]. However, it has drawbacks arising from the fixed Prandtl number ($Pr = 1$) and fixed ratio between the kinematic and bulk viscosities. In order to overcome this, the multiple-relaxation-time (MRT) lattice Boltzmann equation has been developed by d'Humieres [21]. Lallemand and Luo investigated the stability of a MRT lattice Boltzmann equation [22]. Since then, it has attracted more and more attention.

In this paper, the lattice Boltzmann method on a nine speed square lattice (D2Q9) using both MRT and BGK collision terms has been applied to shallow water equations. The advection–diffusion equation is also solved by the lattice Boltzmann method with a D2Q5 lattice model for solute transport. The methods have been applied to three cases: side discharge, shallow recirculation flow and flow in a harbour. The results for velocity fields, temperature fields and concentration distribution have been compared with corresponding experimental data. Furthermore, the relative performance of BGK-LBM and MRT-LBM has been investigated.

2. Governing equations

The two-dimensional shallow water equations and the advection–diffusion equation can be expressed as:

$$\frac{\partial h}{\partial t} + \frac{\partial (hu_j)}{\partial x_j} = 0 \tag{1}$$

$$\frac{\partial hu_i}{\partial t} + \frac{\partial (hu_iu_j)}{\partial x_j} = -g\frac{\partial}{\partial x_i}\left(\frac{h^2}{2}\right) + \nu\frac{\partial^2 (hu_i)}{\partial x_j \partial x_j} + F_i \tag{2}$$

$$\frac{\partial (hc)}{\partial t} + \frac{\partial (u_ihc)}{\partial x_i} = \frac{\partial}{\partial x_i}\left(D_i\frac{\partial (hc)}{\partial x_i}\right) + S_c \tag{3}$$

where the subscripts i and j are space direction indices and the Einstein summation convention is used, t is time, ν is the kinematic viscosity, C is the depth-averaged concentration, D_i is the dispersion coefficient in direction i, S_c is the depth-averaged source term, h is

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water depth, u_i is velocity, x_i stands for either x or y in direction i or j, F_i is the body force per unit mass in direction i and can be expressed by $F_i = -gh\frac{\partial z_b}{\partial x_i} - \frac{\tau_{bi}}{\rho}$. The bed shear stress τ_{bi} in i direction is given by $\tau_{bi} = \rho C_b u_i \sqrt{u_i u_j}$, in which, $C_b = \frac{gn_b^2}{h^{1/3}}$ with Manning coefficient n_b for bed roughness.

3. Lattice Boltzmann models

3.1. Lattice Boltzmann equation for shallow water equations

If the 9-speed square lattice D2Q9 model is adopted, the lattice Boltzmann equations with BGK [23] and MRT [18] for shallow water equations are as follows:

BGK:

$$f_\alpha(x + e_\alpha \Delta t, t + \Delta t) - f_\alpha(x,t) = -\frac{1}{\tau}(f_\alpha(x,t) - f_\alpha^{eq}(x,t)) + \frac{\Delta t}{6e^2} e_{\alpha i} F_i \tag{4}$$

MRT:

$$f_\alpha(x + e_\alpha \Delta t, t + \Delta t) - f_\alpha(x,t) = -T^{-1} S(m_\alpha(x,t) - m_\alpha^{eq}(x,t)) + \frac{\Delta t}{6e^2} e_{\alpha i} F_i \tag{5}$$

where $m = Tf$, $f = T^{-1}$ and S is the relaxation matrix in the moment space, $S = \mathrm{diag}(s1, s2, s3, s4, s5, s6, s7, s8, s9)$, T is the transformation matrix defined by [18]

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 2 & -1 & 2 & -1 & 2 & -1 & 2 & -4 \\ -2 & 1 & -2 & 1 & -2 & 1 & -2 & 1 & -4 \\ 1 & 1 & 0 & -1 & -1 & -1 & 0 & 1 & 0 \\ -2 & 1 & 0 & -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 1 & 0 & -1 & 2 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

f_α is the distribution function of particle, f_α^{eq} is the local equilibrium distribution function, $e = \Delta x/\Delta t$, Δx is the lattice size, Δt is the time step, $e_{\alpha i}$ is the particle velocity in link α. For the 9-speed square lattice shown in Fig. 1, each particle moves one lattice unit at its velocity along the eight links represented with number 1–8 and 0 indicates a particle at rest with zero speed.

The particle velocity vector is

$$e_\alpha = \begin{cases} (0,0), & \alpha = 0 \\ e\left[\cos\frac{(\alpha-1)\pi}{2}, \sin\frac{(\alpha-1)\pi}{2}\right], & \alpha = 1,3,5 \\ \sqrt{2}e\left[\cos\frac{(\alpha-1)\pi}{2}, \sin\frac{(\alpha-1)\pi}{2}\right] & \alpha = 2,4,6,8 \end{cases} \tag{6}$$

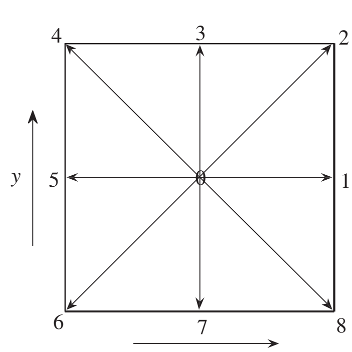

Fig. 1. Nine-speed square lattice (D2Q9) in the horizontal plane.

The local equilibrium distribution function f_α^{eq} is defined as

$$f_\alpha^{eq} = \begin{cases} h - \frac{5gh^2}{6e^2} - \frac{2h}{3e^2} u_i u_i, & \alpha = 0 \\ \frac{gh^2}{6e^2} + \frac{h}{3e^2} e_{\alpha i} u_i + \frac{h}{2e^4} e_{\alpha i} e_{\alpha j} u_i, u_j - \frac{h}{6e^2} u_i u_i, & \alpha = 1,3,5,7 \\ \frac{gh^2}{24e^2} + \frac{h}{12e^2} e_{\alpha i} u_i + \frac{h}{8e^4} e_{\alpha i} e_{\alpha j} u_i, u_j - \frac{h}{24e^2} u_i u_i, & \alpha = 2,4,6,8 \end{cases} \tag{7}$$

The equilibrium values of moments m^{eq} are calculated by:

$$m_{1\ldots9}^{eq} = \left(h, 4h + \frac{3gh^2}{e^2} + \frac{3h(u^2+v^2)}{e^2}, 4h - \frac{9gh^3}{2e^2} - \frac{3h(u^2+v^2)}{e^2}, \frac{hu}{e}, -\frac{hu}{e}, \frac{hv}{e}, -\frac{hv}{e}, \frac{h(u^2-v^2)}{e^2}, \frac{huv}{e^2}\right)^T \tag{8}$$

According to this definition, the water depth h and velocity u_i can be obtained by the following equations [23]

$$h = \sum_\alpha f_\alpha \tag{9}$$

$$u_i = \frac{1}{h}\sum_\alpha e_{\alpha i} f_\alpha \tag{10}$$

By using the Chapman–Engkog procedure [23], the shallow water equations can be recovered from Eqs. (4) and (5) with the kinematic viscosity for BGK

$$v = \frac{\Delta t e^2 \left(\tau - \frac{1}{2}\right)}{6}.$$

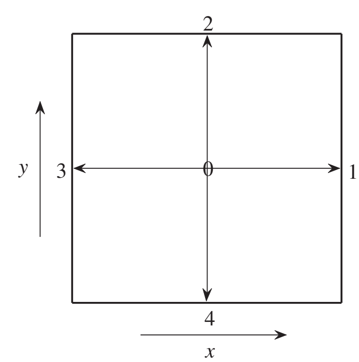

Fig. 2. Five-speed square (D2Q5) lattice in horizontal plane.

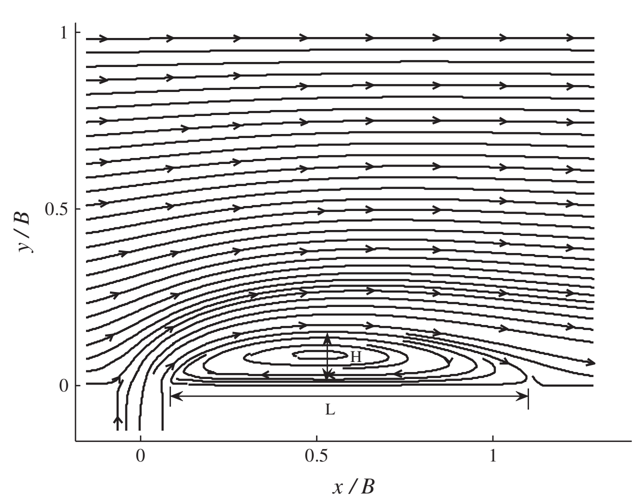

Fig. 3. Predicted streamlines for $M = 0.105$ by MRT-LBM.