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Technical note

Stokes flow in a curved duct – A Ritz method

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ABSTRACT

Fully-developed slow viscous flow in a curved duct of arbitrary curvature is solved by an efficient Ritz variational method. For a duct of rectangular cross section the Ritz results agrees well with those obtained by a Fourier–Bessel expansion. The Ritz method is then applied to the elliptic cross section. The fluid properties for Stokes flow in a curved duct are discussed.

1. Introduction

The viscous flow in a curved duct is fundamental in fluid transport. Hundreds of articles on the various aspects of the flow have been reported [1,2]. But practically all of the literature is concerned with the high or moderate Reynolds number flows, which lead to phenomena such as secondary flow and non-uniqueness.

Due to the miniaturization of fluid apparatus, the flows in small curved ducts are becoming important. Small curved vessels are also common in the microcirculation. Typical Reynolds numbers encountered are 10^{-3} or lower, and the Stokes equation is adequate to describe the flow. There are several consequences of very low Reynolds numbers. Firstly secondary flow, of order Reynolds number, is unimportant. Secondly the entrance effects are limited to less than one width, and the fully developed state is rapidly established. Thus the fully developed results can be useful even for short segments of a curved duct.

Even for Stokes flow, the theoretical analysis of the flow in a curved duct is difficult. One can use Dean's orthogonal coordinates [1], but the resulting equation is not separable, and aside from full numerical integration, only perturbations for a slightly curved tube have been done [3–5]. There seems to be few other relevant literature. In this note we shall present a powerful Ritz method for treating Stokes flow in a curved duct of any cross section and the curvature need not be small.

2. Stokes flow in a curved duct of rectangular cross section

Fig. 1 shows the cross section of the curved duct. Let the centroid of the cross section of the curved duct be on an arc of radius R. We normalize all lengths by R, the velocity by $-RG/\mu$, where G is the azimuthal pressure gradient and μ is the fluid viscosity. The fully developed Stokes equation in cylindrical coordinates (r, θ, z) is

$$\upsilon_{rr} + \frac{1}{r}\upsilon_r - \frac{1}{r^2}\upsilon + \upsilon_{zz} = -\frac{1}{r} \tag{1}$$

where υ is the azimuthal velocity. The boundary condition is that $\upsilon = 0$ on the duct wall. The only analysis for Stokes flow in a curved duct seems to be due to Wang [6] where the cross section is a rectangle of $2bR$ by $2aR$ as in Fig. 1. We shall briefly present a simpler Fourier–Bessel solution for the no-slip case, which will be compared to our Ritz results later.

Let

$$\upsilon = \sum_{n=1}^{\infty} \cos(\beta_n z) f_n(r), \quad \beta_n = \left(n - \frac{1}{2}\right)\frac{\pi}{b} \tag{2}$$

which satisfies the boundary conditions at $z = \pm b$. Now expand unity

$$1 = \sum_{n=1}^{\infty} A_n \cos(\beta_n z), \quad A_n = \frac{2(-1)^{n+1}}{b\beta_n} \tag{3}$$

Eq. (1) gives

$$f_n'' + \frac{1}{r} f_n' - \frac{1}{r^2} f_n - \beta_n^2 f_n = -\frac{A_n}{r} \tag{4}$$

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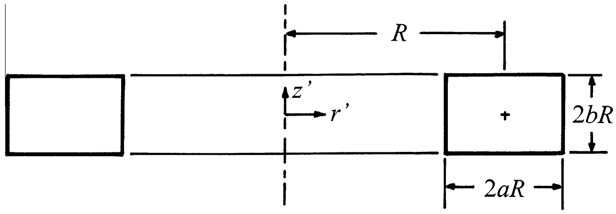

Fig. 1. Cross section of a curved duct and the coordinate system. The dashed line (z-axis) is the symmetry axis about which the cross section is rotated.

The general solution is

$$f_n(r) = \frac{A_n}{\beta_n^2 r} + C_{1n}K_1(\beta_n r) + C_{2n}I_1(\beta_n r) \tag{5}$$

Here K_1 and I_1 are modified Bessel functions. The boundary conditions at $r = 1 \pm a$ are

$$f_n(1-a) = 0, \quad f_n(1+a) = 0 \tag{6}$$

giving

$$\begin{aligned} C_{1n} &= A_n\{(1+a)I_1[(1+a)\beta_n] - (1-a)I_1[(1-a)\beta_n]\}/D_n \\ C_{2n} &= -A_n\{(1+a)K_1[(1+a)\beta_n] - (1-a)K_1[(1-a)\beta_n]\}/D_n \\ D_n &= (1-a^2)\beta_n^2\{I_1[(1-a)\beta_n]K_1[(1+a)\beta_n] \\ &\quad - I_1[(1+a)\beta_n]K_1[(1-a)\beta_n]\} \end{aligned} \tag{7}$$

The flow rate, normalized by R^3G/μ, is then

$$Q = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\beta_n} S_n \tag{8}$$

where

$$\begin{aligned} S_n &= \int_{1-a}^{1+a} f_n(r)dr \\ &= \beta_n\{2A_n\beta_n(\tanh^{-1}a) - C_2\langle I_0[(1-a)\beta_n] - I_0[(1+a)\beta_n]\rangle \\ &\quad + C_1\langle K_0[(1-a)\beta_n] - K_0[(1+a)\beta_n]\rangle\} \end{aligned} \tag{9}$$

The average or mean velocity is

$$V = \frac{Q}{4ab} \tag{10}$$

If the width b is infinite (a slit), the z dependence is absent and the form of the solution is different. Eq. (1) gives

$$\upsilon = -\frac{1}{2} r\ln r + C_3 r + C_4\frac{1}{r} \tag{11}$$

The boundary conditions give

$$C_3 = -\frac{(1-a)^2}{8a}\ln\left(\frac{1-a}{1+a}\right) + \frac{1}{2}\ln(1+a) \tag{12}$$

$$C_4 = \frac{(1-a^2)^2}{8a}\ln\left(\frac{1-a}{1+a}\right) \tag{13}$$

The average velocity is

$$V = \frac{1}{2a}\int_{1-a}^{1+a}\upsilon\, dr = \frac{1}{16a^2}\{4a^2 - (1-a^2)^2\left[\ln\left(\frac{1-a}{1+a}\right)\right]^2\} \tag{14}$$

3. The Ritz method

We present the Ritz method which can be applied to any cross section. The Ritz or Rayleigh–Ritz variational method [7,8] has been used extensively in vibration of plates and membranes, but not as often in fluid mechanics, After some work, we find Eq. (1) is equivalent to minimizing the following integral over the cross sectional area.

$$J = \iint (r\upsilon_r^2 + \upsilon^2/r + r\upsilon_z^2 - 2\upsilon)dzdr \tag{15}$$

This can be verified by using Euler's condition for minimizing a functional [7]. Let $s = r - 1$ and let

$$g(s,z) = 0 \tag{16}$$

describe the tube wall. Let the solution be represented by the series

$$\upsilon = \sum_{n=1}^{\infty} c_i\varphi_i(s,z) \tag{17}$$

where c_i are coefficients to be determined, and φ_i is a complete set of functions which satisfy the boundary conditions. The necessary condition for minimal J is

$$\frac{\partial J}{\partial c_i} = 0 \tag{18}$$

which can be shown to be equivalent to

$$\sum A_{ij}c_j = \sum B_i \tag{19}$$

where

$$A_{ij} = \iint [(s+1)(\varphi_{is}\varphi_{js} + \varphi_{iz}\varphi_{jz}) + \frac{1}{s+1}\varphi_i\varphi_j]\,dzds \tag{20}$$

$$B_i = \iint \varphi_i\, dzds \tag{21}$$

Then the linear algebraic Eq. (19) is inverted for the coefficients c_i. The flow rate is simply

$$Q = \iint \upsilon\, dzds = \sum c_iB_i \tag{22}$$

and the average velocity is

$$V = \frac{Q}{\iint dzds} \tag{23}$$

We illustrate by computing the Stokes flow through the curved rectangular duct studied previously. The boundary is bounded by

$$g = (z^2 - b^2)(s^2 - a^2) = 0 \tag{24}$$

Consider the set of polynomials

$$\{\varphi_i\} = g(z,s)\{1, s, s^2, z^2, s^3, sz^2, s^4, s^2z^2, z^4, s^5, s^3z^2, sz^4, \ldots\} \tag{25}$$

where due to symmetry, only the even powers of z are used. The number of terms can be taken as 4, 6, 9, 12, 16, etc., retaining the highest homogeneous powers. Eqs. (20) and (21) are evaluated by integrating analytically with respect to y then numerically with respect to x (Mathematica adaptive recursion library program with a relative error of 10^{-8}). The coefficients are then found by Eq. (19). Table 1 shows a comparison of the two methods. Both are accurate and efficient.

The results for other dimensions are given in Table 2. Both methods agree within 0.1%. The $b = \infty$ results are from Eq. (14). Typical constant velocity lines are shown in Fig. 2.

Table 1
Typical convergence for Bessel function solution and Ritz solution for a rectangular duct ($a = b = 0.5$).

Terms used	4	6	9	12	16	20
Eq. (10)	0.03541	0.03543	0.03544	0.03544	0.03544	0.03544
Eq. (22)	0.03532	0.03537	0.03543	0.03543	0.03544	0.03544